# 18.786. (Spring 2014) Problem set # 4 (due Tue Mar 11)

* This problem set is lengthier but mainly due to explanation of notation and definitions.

1. (Extending coefficients) Let $L' \supset L \supset \mathbb{Q}_\ell$ be finite extensions. Write $\mathcal{O}'$ (resp. $\mathcal{O}$) for the ring of integers of $L'$ (resp. $L$) and $\mathbb{F}'$ (resp. $\mathbb{F}$) for its residue field. Let $\Gamma$ be a profinite group satisfying Hyp($\Gamma$) (i.e. the condition in the first paragraph of [Gee, 3.1]), $\overline{\rho} : \Gamma \to GL_n(\mathbb{F})$ a continuous representation. Denote by $\overline{\rho}' := \overline{\rho} \otimes_{\mathbb{F}} \mathbb{F}' : \Gamma \to GL_n(\mathbb{F}')$ the extension of coefficients from $\overline{\rho}$ (in other words, the composition $\overline{\rho}$ with $GL_n(\mathbb{F}) \subset GL_n(\mathbb{F}')$). Prove that there is a canonical isomorphism (in the category $\mathcal{C}_{\mathcal{O}'}$)
$$R^{\square}_{\overline{\rho}'} \simeq R^{\square}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}',$$
where $R^{\square}_{\overline{\rho}'}$ (resp. $R^{\square}_{\overline{\rho}}$) is the universal lifting ring for $\overline{\rho}'$ in $\mathcal{C}_{\mathcal{O}'}$ (resp. $\overline{\rho}$ in $\mathcal{C}_{\mathcal{O}}$).[1]

2. (Compare with [Gee, Exercise 3.11]) Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_l$, and $\overline{\rho} : \Gamma \to GL_{\mathbb{F}}(\overline{V})$ a continuous representation, where $\Gamma$ is a profinite group satisfying Hyp($\Gamma$). Assume that $\overline{\rho}$ is absolutely irreducible (so that the universal deformation ring $R^{\mathrm{univ}}_{\overline{\rho}} \in \mathcal{C}_{\mathcal{O}}$ exists; its unique maximal ideal is denoted $\mathfrak{m}_{R^{\mathrm{univ}}_{\overline{\rho}}}$). Construct natural maps between the following sets and show that they are bijections.
   (a) $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\mathrm{univ}}_{\overline{\rho}}}/(\mathfrak{m}^2_{R^{\mathrm{univ}}_{\overline{\rho}}}, \lambda), \mathbb{F})$
   (b) $\mathrm{Hom}_{\mathcal{O}}(R^{\mathrm{univ}}_{\overline{\rho}}, \mathbb{F}[\epsilon]/(\epsilon^2))$
   (c) $H^1(\Gamma, \mathrm{ad}\overline{\rho})$
   (d) $\mathrm{Ext}^1_{\mathbb{F}[\Gamma]}(\overline{\rho}, \overline{\rho})$

   Here $H^1$ and $\mathrm{Ext}^1$ denote *continuous* cohomology and extension classes, resp. For $H^1$ this means that 1-cocycles are required to be continuous maps.[2] We recall the definition of $\mathrm{Ext}^1_{\mathbb{F}[\Gamma]}(\overline{\rho}, \overline{\rho})$ here: it's the equivalence classes of extensions
   $$0 \to \overline{\rho} \to \xi \to \overline{\rho} \to 0,$$
   where $\xi$ is a continuous $\mathbb{F}[\Gamma]$-module and the maps are $\mathbb{F}[\Gamma]$-module morphisms.[3] The two extensions $\xi$ and $\xi'$ are equivalent if there is a commutative diagram
   $$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\rho} & \longrightarrow & \xi & \longrightarrow & \overline{\rho} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{\rho} & \longrightarrow & \xi' & \longrightarrow & \overline{\rho} & \longrightarrow & 0 \end{array}$$
   such that the vertical maps are isomorphisms of $\mathbb{F}[\Gamma]$-modules.

3. In the proof of [Gee, Lem 3.13] we encounter the following situation. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}[[\underline{x}]] = \mathcal{O}[[x_1, ..., x_d]]$, and $J := \ker \phi$ as in the lemma. Then $\mathfrak{m}J \subset J$, and $\rho = \rho^{\square}_{\overline{\rho}} : \Gamma \to GL_n(\mathcal{O}[[\underline{x}]]/J)$.[4] For each $\gamma \in \Gamma$ choose any lift $\widetilde{\rho(\gamma)}$ of $\rho(\gamma)$ via the surjection
   $$GL_n(\mathcal{O}[[\underline{x}]]/\mathfrak{m}J) \twoheadrightarrow GL_n(\mathcal{O}[[\underline{x}]]/J).$$
   Now let $f \in \mathrm{Hom}_{\mathbb{F}}(J/\mathfrak{m}J, \mathbb{F})$. We already know from class that
   $$c_f(\gamma, \delta) := f\left(\widetilde{\rho(\gamma\delta)}\,\widetilde{\rho(\delta)}^{-1}\,\widetilde{\rho(\gamma)}^{-1} - \mathbf{1}_n\right) \in M_n(\mathbb{F}), \quad \forall \gamma, \delta \in \Gamma$$
   is a continuous 2-cocycle in $Z^2(\Gamma, \mathrm{ad}\overline{\rho})$. Write $J_f$ for the kernel of the composite map $J \to J/\mathfrak{m}J \xrightarrow{f} \mathbb{F}$. Show that

[1] The same argument will show that $R^{\mathrm{univ}}_{\overline{\rho}'} \simeq R^{\mathrm{univ}}_{\overline{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$ when $\overline{\rho}$ is absolutely irreducible (equivalently when $\overline{\rho}'$ is absolutely irreducible). Of course you need not write this up. A variant of this isomorphism also exists when the determinant is fixed in the lifting/deformation problem, cf. [Gee, 3.18].

[2] The notion of *continuous* 2-cocycles and *continuous* $H^2$-cohomology classes is defined in the same way below.

[3] As usual the underlying $\mathbb{F}$-vector spaces are equipped with discrete topology.

[4] My $\Gamma$ is Gee's $G$.

(a) $c_f$ gives a well-defined element $[c_f] \in H^2(\Gamma, \mathrm{ad}\rho)$, i.e. it is independent of the choices of $\widetilde{\rho(\gamma)}$ above.

(b) $f \mapsto [c_f]$ is $\mathbb{F}$-linear.

(c) $[c_f] \in H^2(\Gamma, \mathrm{ad}\rho)$ is trivial if and only if there exist choices of $\widetilde{\rho(\gamma)}$ for all $\gamma \in \Gamma$ such that the map $\Gamma \to GL_n(\mathcal{O}[[\underline{x}]]/J_f)$ induced by $\gamma \mapsto \widetilde{\rho(\gamma)}$ is a homomorphism.

Feel free to look up [Ser-LF], [AW], Serre's "Galois cohomology" book, etc for the definition of 2-cocycles, $H^2$, etc.

# References

[Gee] Toby Gee, *Modularity lifting theorems – Notes for Arizona winter school*, draft, http://www2.imperial.ac.uk/~tsg.