# 18.786. (Spring 2014) Problem set # 1 (due Thu Feb 13)

1. Prove that a profinite group $\Gamma$ (with profinite topology) is compact, Hausdorff, and totally disconnected. (The last condition means that every connected proper subset of $\Gamma$ has at most one element.)

2. [Gee, Exercise 2.2] Let $F$ be a field. Show that a continuous homomorphism $\rho : \mathrm{Gal}(\overline{F}/F) \to GL_n(\mathbb{C})$ factors through $\mathrm{Gal}(E/F)$ for a finite extension $E/F$ in $\overline{F}$ and that $\rho$ has image in $GL_n(\overline{\mathbb{Q}})$ possibly after conjugation by an element of $GL_n(\mathbb{C})$. (Here $\overline{\mathbb{Q}}$ is viewed as the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.)

3. Complete the proof of the Brauer-Nesbitt theorem. Let's recall the setup. Let $k$ be a field, $\Gamma$ be a group, $V_j$ be an $n_j$-dimensional vector space over $k$, and $\rho_j : \Gamma \to GL_k(V_j)$ be *semisimple* representations of $\Gamma$, where $j = 1, 2$. If
$$\forall \gamma \in \Gamma, \quad \det(1 - \rho_1(\gamma)T) = \det(1 - \rho_2(\gamma)T)$$
(i.e. the characteristic polynomials are the same for the two representations) then your problem is to prove that $\rho_1 \simeq \rho_2$. Freely use the following lemma (proved in class):

   **Lemma 0.1.** *Let $r \geq 1$. Let $R$ be an associate $k$-algebra (which may not be commutative) and $M_1, ..., M_r$ be simple (left) $R$-modules which are mutually non-isomorphic and finite dimensional over $k$. Then there exist $e_1, ..., e_r \in R$ such that the multiplication map $e_i$ is the identity map on $M_i$ and the zero map on $M_j$ for all $j \neq i$.*

4. Let $n \in \mathbb{Z}_{\geq}1$. Let $\ell$ be a prime. Let $L$ be a finite extension of $\mathbb{Q}_\ell$, $\mathcal{O}_L$ be its ring of integers, and put $\Lambda := \mathcal{O}_L^n$.

   - Show that $1+\ell^m \cdot \mathrm{End}_{\mathcal{O}_L}(\Lambda)$ is a pro-$\ell$-subgroup of $GL_{\mathcal{O}_L}(\Lambda)$ for each $m \in \mathbb{Z}_{\geq 1}$. (Here the subscript $\mathcal{O}_L$ indicates that one considers $\mathcal{O}_L$-*linear* endomorphisms and automorphisms, respectively.)
   - Let $\Gamma$ be a profinite group whose order is "prime to $\ell$" in the sense that every finite group quotient of $\Gamma$ has prime-to-$\ell$ order. Suppose that $\rho : \Gamma \to GL_{\mathcal{O}_L}(\Lambda)$ is a continuous homomorphism whose image is contained in $1 + \ell \cdot \mathrm{End}_{\mathcal{O}_L}(\Lambda)$. Then prove that $\rho$ is the trivial representation.

## References


[Gee] Toby Gee, *Modularity lifting theorems - notes for arizona winter school*, draft, `http://www2.imperial.ac.uk/~tsg`.