Stochastic differential equations & applications in finance

Consider classical ODE such as a population N of bacteria

$$\frac{dN}{dt} = \underbrace{a(t)}_{\text{growth rate}} N(t) \quad , N(0) = N_0$$

What if $a(t)$ is subject to noise, i.e. $a(t) = r(t) + \xi$ ← noise

(ξ: only known probab. density function)

$\Rightarrow N(t) = ?$ How do we solve these equations?

In fact, does $\frac{dN}{dt}$ in the sense of a usual derivative even exist if noise is added?

Let's look at this topic by means of an example: random walks (1D)

Consider 1D unbiased RW in position X. At each step i, we randomly choose to either go left or right by distance l. After N steps, then, the position is (starting at x_0):

$$X_N = x_0 + l \sum_{i=1}^{N} s_i \quad , \quad s_i \in \{-1, 1\}, \quad p(s_i = 1) = p(s_i = -1) = 1/2$$

The expectation value $E(s_i) = 1/2 \cdot (+1) + 1/2 (-1) = 0$

$$i \neq j: \quad E(s_i s_j) = \tfrac{1}{2}(1) \cdot \tfrac{1}{2}(-1) + \tfrac{1}{2}(-1) \cdot \tfrac{1}{2}(1) + \tfrac{1}{2}(1) \cdot \tfrac{1}{2}(1) + \tfrac{1}{2}(-1) \cdot \tfrac{1}{2}(-1) = 0 \quad i \neq j$$

$$E(s_i s_i) = E(s_i^2) = (-1)^2 \cdot \tfrac{1}{2} + 1^2 \cdot \tfrac{1}{2} = 1$$

$$\Rightarrow E(s_i s_j) = \delta_{ij}$$

Then:

$$E(X_N) = x_0 + l \sum E(s_i) = x_0$$

$$E(X_N^2) = E([x_0 + l \sum E(s_i)]^2)$$

$$= E\left(x_0^2 + 2x_0 \ell \sum_i^N S_i + \ell^2 \sum_i^N \sum_j^N S_i S_j\right)$$

$$= x_0^2 + 2x_0 \cdot 0 + \ell^2 \sum_i^N \sum_j^N \delta_{ij}$$

$$= x_0^2 + \ell^2 N$$

Therefore, the variance of this process is

$$\mathrm{var} = E([X_N - E(X_N)]^2) = E(X_N^2 - 2X_N E(X_N) + E(X_N)^2)$$

$$= E(X_N^2) - 2E(X_N) \cdot E(X_N) + E(X_N)^2$$

$$= \underbrace{E(X_N^2)}_{x_0^2 + \ell^2 N} - \underbrace{E(X_N)^2}_{0}$$

$\Rightarrow$ Variance grows like N, ie $\sim \ell^2 N$

Now let's look at the continuum limit of this process:

For simplicity, $x_0 = 0$, and we consider an <u>even</u> number of steps $N = t/\tau$ (τ: time required for a single step)

Then, the probability $P(N,K)$ to be at the position $\frac{x}{\ell} = K > 0$ after N steps is given by the binomial coefficient:

~~$P(N,K)$~~

To end up at $K > 0$ after N steps, the walker must go $n_1 = K + n_2$ steps to the right and n_2 steps to the left, and $n_1 + n_2 = N$. Therefore,

$$n_1 = \tfrac{1}{2}(N+K) \text{ jumps to the right} \qquad (*)$$
$$n_2 = \tfrac{1}{2}(N-K) \text{ jumps to the left}$$

Since it doesn't matter in which order we do these n_1, n_2 steps, we have $\frac{N!}{\left(\frac{N+K}{2}\right)! \left(\frac{N-K}{2}\right)!}$ possibilities; ~~each with probability~~ $\frac{1}{2}$

$$\frac{N!}{n_1! \cdot n_2!} = \frac{N!}{n_1!(N-n_1)!} \overset{(*)}{=} \frac{N!}{\frac{1}{2}(N+K)! \cdot \frac{1}{2}(N-K)!}$$

Since each of the N steps has prob. 1/2 (no matter if we go left or right), the prob. becomes

$$p(N,K) = \left(\tfrac{1}{2}\right)^N \frac{N!}{(N+K)/2\,!\cdot\frac{N-K}{2}!} = \left(\tfrac{1}{2}\right)^N \underbrace{\binom{N}{\frac{N+K}{2}}}_{\text{binomial coefficients}}$$

One can show that, by defining $p(t,x) := \frac{P(N,K)}{2\ell} = \frac{P(t/\tau,\, x/\ell)}{2\ell}$

the limit $\tau, \ell \to 0$ with $D = \frac{\ell^2}{2\tau}$ constant, leads to the Gaussian distribution

$$p(x,t) = \sqrt{\frac{1}{4\pi D t}}\, e^{-\frac{x^2}{4Dt}} \qquad (**)$$

We already have seen that $(**)$ solves the heat eq. with $\delta(x)$ as I.C., i.e

$$p_t = D\, p_{xx}$$

Note that we again find $E(X(t)^2) = \int dx\, x^2 p(x,t) = \dots = 2Dt$
i.e linear growth of variance in time, as we have seen before.
We also see that a random walk „solves" in a probabilistic way the diffusion equation.

For biased random walks, the walker goes to left with prob. λ and to the right with prob. ρ, or it may remain at current position with probability $1-\rho-\lambda$. A similar calculation then shows that

$$\partial_t p = + u\, \partial_x p + D\, \partial_{xx} p \qquad [\text{advection diffusion eq!}]$$

with drift $u = -(\rho-\lambda)\frac{\ell}{\tau}$, $D = (\rho+\lambda)\frac{\ell^2}{2\tau}$

~~In oth~~

We see that random processes fulfill ordinary PDE's in a probabilistic sense. However, the process might be nonlinear, making the derivation of a analytical expression for the pdf dificult/impossible. Need a way to simulate random processes. FD in the usual sense is dificult, since we aren't solving a differential equation. Still, the continuum limit for the pdf suggests that some well-defined evolution equation exists. These are the SDE.

In the context of our example, of the adv.-dif. eq., we note that we could also write it as

$$\dot{X} \text{"="} u + \text{„noise"}$$

where the noise is a random walk with equal prob. to left & right.

How would we integrate this equation?

In fact, we can write the advect. diff. eq. in the differential form

$$dX(t) = u\,dt + \sqrt{2D}\,dB(t) \qquad (***)$$

with $dX(t) = X(t+dt) - X(t)$, $dB = B(t+dt) - B(t)$ the increment of the Brownian (random) motion $B(t)$. $B(t)$ is defined as

- $B(0) = 1$
- $B(t)$ is stationary, i.e. $B(t) - B(s)$ has the same distribution as $B(t-s)$
- $B(t)$ has indep. increments, i.e. $B(t_n) - B(t_{n-1})$, $B(t_{n-1}) - B(t_{n-2})$, ... are all indep. distributed.
- $B(t)$ has Gaussian distribution with variance t for all t
- $B(t)$ is continuous

Often, $(***)$ is written as $\dot{X}(t) = u + \sqrt{2D}\,\xi(t)$ although the derivative $\xi(t) = \frac{dB}{dt}$ is mathematically not well-defined.

To fullfill the properties of B, ξ then is characterized by

$$\langle \xi(t) \rangle = 0 \quad , \quad \langle \xi(t)\xi(s) \rangle = \delta(t-s)$$

"Gaussian white-noise"

To see that Eq. (***) gives a adv. diff. equation one needs to derive the Fokker-Planck equations, but setting $u=0$, we can qualitatively see that the remaining eq. is a random walk/diffusion:

$$dX = \sqrt{2D}\, dB(t) \qquad \text{(iv)}$$

Since $B(t)$ has indep. increments and $B(t)$ is stationary and Gaussian with variance t, it follows that

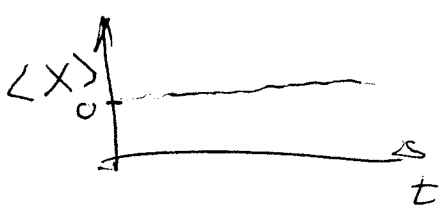
$$B(t) = B(t-\delta t) + dB(t) \overset{\text{stationary } B}{\Rightarrow} dB \text{ has same distrib. as } B(\delta t)$$

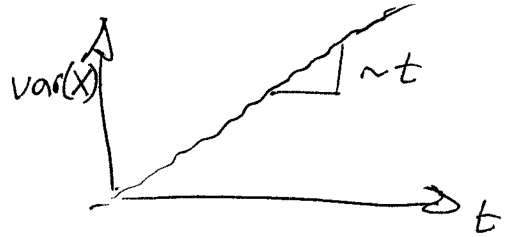
where each $dB(t)$ is a random variable of the form $\sqrt{\delta t}\,\mathcal{N}(0,1)$ with $\mathcal{N}(0,1)$ the standard normal dist. (mean: 0, $\sigma = 1$).

We can therefore numerically integrate (iv) as

$$X(t+\delta t) = X(t) + \sqrt{2D\,\delta t}\,\mathcal{N}(0,1)$$

Doing this over many samples of random paths, we find





as in a random walk, but note that this process is contin. in the sense that δt can be chosen arbitrarily, like in a PDE!

Slides / Matlab

The same numerical method with drift gives

$$X(t+\delta t) = X(t) + u\,\delta t + \sqrt{2D\delta t}\,\mathcal{N}(0,1)$$

Note that if u and D depend on X, then different integrations can be performed:

$$X(t+\delta t) = X(t) + u(X(t))\,\delta t + \sqrt{2D(X(t))\,\delta t}\,\mathcal{N}(0,1)$$

"stochastic Euler (forward). Note ↑ at old time!

For ODE's (i.e. if $B(t)$ was a deterministic function) it wouldn't matter if we evaluate X at t or $t+\delta t$ in $u(X)$ and $D(X)$. This is because the upper and lower Rieman sums converge to the same value as $\delta t \to 0$. However, if $B(t)$ is a rapidly changing stochastic process, the upper and lower Rieman sums do not generally converge to the same value. The type of sum/integral convention chosen is therefore relevant. The above integral is called the Ito integral.

If we would choose

$$X(t+\delta t) = X(t) + u(X(t+\delta t))\,\delta t + \sqrt{2D(X(t+\delta t))\,\delta t}\,\mathcal{N}(0,1)$$

then the numerical scheme resembles the implicit Euler scheme. This is the Backwards-Ito integral, and it corresponds to an upper Rieman sum. As such, it does not converge to the same solution as the Forward Euler / Ito integral, in contrast to ODE's !!!

~~Soluti~~

Quality of numerical solutions

To investigate the „error" and convergence properties, we make an example based on a ~~two~~ SDE with known analytical solution.

We will to this end look at the SDE

$$d X(t) = \lambda X(t) dt + \mu X(t) dB \quad , \quad X(0) = X_0 \qquad (1)$$

(like $u = \lambda X$, $\mu = D X$ in previous eq.)

The solution to this SDE is (we are not getting into stochastic calculus here...)

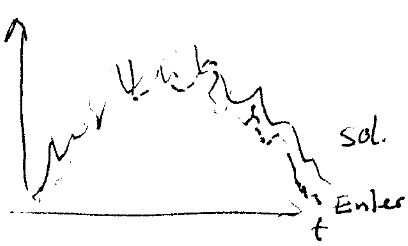
$$X(t) = X(0) e^{(\lambda - \frac{1}{2}\mu^2) t + \mu B(t)} \qquad (2)$$

To test the numerical scheme, we can use Euler to integrate (1) and compare it with (2) at each time step $t = t_1, t_2, \dots$,

We therefore construct a path $B(t)$ at times $t_i = i \cdot \delta t$, with $\delta t = 2^{-8}$ by summing up the increments $dB(t)$. $\to B(t) \to X(t)$ [(2)].

Similarly, we integrate Eq (1) using, e.g. $\Delta t = 4 \delta t$ allowing us to ~~see~~ how Δt affects accuracy.

see slides



accuracy.m

How well does it converge?

Since we're dealing with stochastic processes, ~~the~~ one way of defining convergence is to say: X_n converges to $X(T)$ if there is a const. C such t.

$$E(|X_n - X(T)|) \leq C \Delta t^{\gamma} \qquad \text{for any fixed } T = n \cdot \Delta t$$

One can show that the Euler scheme has $\gamma = 1/2$.
The above convergence criterium is called <u>strong convergence</u>.
We are not proving this but look at numerical experiments.

→ <u>slides</u>

Stochastic chain rule

In the deterministic case, if you have a function $F(X)$, its derivative is

$$\frac{dF}{dt} = \frac{dF}{dX}\frac{dX}{dt}$$ by the chain rule.

What if $dX = u\,dt + \sqrt{2D}\,dB$?

$$\Rightarrow dF(X(t)) = F(X(t+dt)) - F(X(t))$$
$$= F'(X(t))dX + \frac{1}{2}F''(X(t))dX^2 + \dots$$
$$= F'(X(t))dX + \frac{1}{2}F''(X(t))\left[u\,dt + \sqrt{2D}\,dB\right]^2 + \dots$$
$$= F'(X(t))dX + DF''(X(t))\,dB^2 + O(dt^{3/2})$$ since $dB \sim dt^{1/2}$

Therefore, to leading order in dt:

$$dF(X(t)) = F'(X(t))dX + DF''(X(t))dt$$
$$= \left[u F'(X(t)) + DF''(X(t))\right]dt + \underbrace{F'(X(t))\sqrt{2D}\,dB(t)}_{!!!}$$

For a more general SDE

$$dX = f(X(t))dt + g(X(t))dB(t)$$

the „stochastic chain rule" for a function $F(X)$ is:

$$\boxed{dF(X(t)) = \left[f(X(t))F'(X(t)) + \frac{1}{2}g(X(t))^2 F''(X(t))\right]dt + g(X(t))F'(X(t))dB(t)}$$

<u>Itô's Formula</u>