Modified equations

So far, we investigated accuracy of FD via the local „truncation" error DF_{n+1}. The idea was to see how well the true solution of the PDE satisfies the FD equation ⇝ accuracy.

Now: We ask the question: Given a FD scheme. Is there a PDE $u_t = \ldots$ ~~which~~ for which our numerical approximation $u_{i+1} = \ldots$ is actually the exact solution?

Or a bit more relaxed: Is there a PDE which is solved exactly by our FD scheme to some order in Δt that is higher than the accuracy for the original PDE approximation.

These equations are called modified equations.

They are useful because ~~[illegible]~~ the analyt. solution of the mod. eq. will tell us what the FD scheme will do! This is often easier than analyzing the FD scheme!

Example: Upwind FD, 1-way wave equation $u_t = c u_x$, $c>0$

FD: $$u_{j,n+1} = u_{j,n} + \frac{c\Delta t}{\Delta x}\left(u_{j+1,n} - u_{j,n}\right) \qquad (*)$$

Assume now that there is an analytical, smooth function $v(x,t)$ which agrees exactly at grid points whith $(*)$, i.e.

$$v(x,t+\Delta t) = v(x,t) + \frac{c\Delta t}{\Delta x}\left(v(x+\Delta x,t) - v(x,t)\right)$$

We now expand all v's around (x,t):

$$v(x,t) + v_t(x,t)\Delta t + \tfrac{1}{2}\Delta t^2 v_{tt}(x,t) + \tfrac{1}{6}\Delta t^3 v_{ttt}(x,t) + \dots$$
$$= v(x,t) + \frac{c\Delta t}{\Delta x}\left(v(x,t) + v_x(x,t)\Delta x + \tfrac{1}{2}\Delta x^2 v_{xx}(x,t) + \tfrac{1}{6}\Delta x^3 v_{xxx}(x,t) + \dots - v(x,t)\right)$$

$$= v_t + \tfrac{1}{2}\Delta t\, v_{tt} + \tfrac{1}{6}\Delta t^2 v_{ttt} + \dots - c\left(v_x + \tfrac{1}{2}\Delta x\, v_{xx} + \tfrac{1}{6}\Delta x^2 v_{xxx} + \dots\right) = 0$$

$$\Rightarrow v_t - c v_x = \tfrac{1}{2}\left(c\Delta x\, v_{xx} - \Delta t\, v_{tt}\right) + \tfrac{1}{6}\left(c\Delta x^2 v_{xxx} + \Delta t^2 v_{ttt}\right) + \dots$$

This is the PDE that $v(x,t)$ satisfies exactly.

~~Droppin~~ If we take $\frac{\Delta t}{\Delta x} = \frac{r}{c}$ fixed, then:

$$v_t - c v_x = \underbrace{\tfrac{1}{2}\Delta t\Big(c \underbrace{\tfrac{\Delta x}{\Delta t}}_{\text{fixed}} v_{xx} - v_{tt}\Big)}_{O(\Delta t)} + \underbrace{\dots}_{O(\Delta t^2)} + \dots$$

To lowest order in Δt, v solves the original PDE $v_t - c v_x = 0$

Since this means that we dropped all terms $\sim O(\Delta t)$, we expect that the FD scheme approximates $u_t - c v_x$ to order Δt globally

$\rightarrow$ agrees with global error $O(\Delta t)$ of Upwind FD scheme.

But let's see what happens at next order, so we keep everything $O(\Delta t)$:

$$v_t - c v_x = \tfrac{1}{2}\left(c\Delta x\, v_{xx} - \Delta t\, v_{tt}\right) \qquad (**)$$

Unfortunatelly, this isn't yet in a good form for analytical insight.

Let's take derivative w.r.t. t of (**) :

$$v_{tt} = c v_{xt} + \frac{1}{2}\left(c \Delta x \, v_{xxt} - \Delta t \, v_{ttt}\right)$$

Instead taking derivative w.r.t. x of (**):

$$v_{tx} = c v_{xx} + \frac{1}{2}\left(c \Delta x \, v_{xxx} - \Delta t \, v_{ttx}\right)$$

Combining them gives

$$v_{tt} = c^2 v_{xx} + O(\Delta t)$$

Inserting into (**) we get

$$v_t - c v_x = \frac{1}{2}\left(c \Delta x \, v_{xx} - c^2 \Delta t \, v_{xx}\right) + O(\Delta t^2)$$

$$\Rightarrow \quad v_t - c v_x = \frac{1}{2} c \Delta x \left(1 - \frac{c \Delta t}{\Delta x}\right) v_{xx} + O(\Delta t^2)$$

This is an advection - diffusion equation (see later)

↑ 1-way wave eqn.   ↑ $\sim v_{xx}$ term

<u>Interpretation</u>: Grid values $U_{j,n}$ give a <u>second</u> order accurate interpolation of an advection-diffusion equation.

→ We see that the numerical sol. to $u_t = c u_x$ will advect as it should, but it will <u>smear out</u> due to the diffusive term.
with speed c

⇒ Show <u>figure</u> in <u>slide</u>

Note the diffusion constant $\frac{1}{2} c \Delta x \left(1 - \frac{c \Delta t}{\Delta x}\right) = \frac{1}{2} c \Delta x (1 - r) = D$

⇒ no diffusion when $r = 1$. For $r = 1$, Neumann stab. analysis gave

$$|G|_{discrete} = \left| r e^{ik\Delta x} + (1-r) \right| \overset{r=1}{=} 1 = |G|_{analytical}$$

~~In other words, we can even extract some information~~

Note that $D \geq 0 \iff 0 < r \leq 1$, which was our previous stability criterium!

$\Rightarrow$ Some info about stability of FD can be extracted from behaviour of modified equation!

Modified eq. for Lax-Wendroff

The same approach for Lax-Wendroff gives a modified equation of the form

$$V_t - cV_x = \underbrace{\frac{1}{6} c \Delta x^2 \left[1 - \left(\frac{c\Delta t}{\Delta x}\right)^2\right]}_{D} V_{xxx} \qquad (***)$$

$\rightarrow$ no diffusion, but V_{xxx} leads to dispersion

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Can we understand this oscillatory behavior?

$\Rightarrow$ Solve $(***)$ for plane waves $\sim e^{i(kx+\omega t)}$. Plugging in $Ae^{i(kx+\omega t)}$

$$+i\omega A e^{i(kx+\omega t)} - cik A e^{i(kx+\omega t)} = D(ik)^3 A e^{i(kx+\omega t)}$$

with D defined above

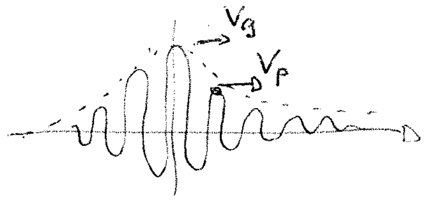
$\Rightarrow \omega - ck = -Dk^3$ $\Rightarrow \omega(k) = -Dk^3 + ck$: dispersion relation of modified equation

$\Updownarrow$ compare:

$\omega(k) = ck$ for 1-way wave equation

Remember from physics:

- Phase velocity of a wave: $V_p = \frac{\omega}{k}$
- Group velocity of a wave: $V_G = \frac{\partial \omega}{\partial k}$



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V_g is velocity of envelope

V_p is velocity of „phase" (high freq. oscillations)

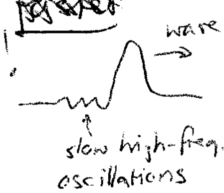
- For 1-way wave equation: $V_p = \frac{\omega}{k} = \frac{ck}{k} = c$

$$V_g = \frac{\partial \omega}{\partial k} = c = V_p$$

independent of k! All k-waves travel by same speed.

Here for the modified equation, we have

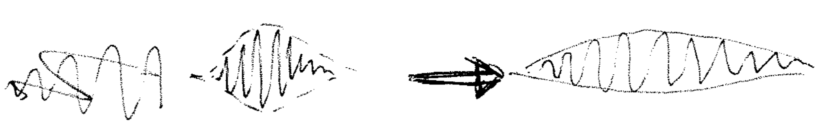
- $V_p(k) = -Dk^2 + c$: since $D \geq 0$ for $0 < r \leq 1$, this means high freq. waves (k large) travel ~~faster~~ slower than low freq. waves!



- $V_g = \frac{\partial \omega}{\partial k} = c - 3Dk^2 = c\left(1 - \frac{1}{2}\Delta x^2\left[1 - r^2\right]k^2\right)$

$\Rightarrow$ also group velocity is always smaller than c for all k's!

From physics it is known that $V_g(k) \neq$ const means that the wave envelope will get distorted over time!



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