One can show that the Euler scheme has $\gamma = 1/2$.
The above convergence criterium is called <u>strong</u> convergence.
We are not proving this but look at numerical experiments.

→ <u>slides</u>

Stochastic chain rule

In the deterministic case, if you have a function $F(X)$, its derivative is

$$\frac{dF}{dt} = \frac{dF}{dX}\frac{dX}{dt} \quad \text{by the chain rule.}$$

What if $dX = u\,dt + \sqrt{2D}\,dB$?

$$\Rightarrow dF(X(t)) = F(X(t+dt)) - F(X(t))$$
$$= F'(X(t))dX + \tfrac{1}{2}F''(X(t))dX^2 + \dots$$
$$= F'(X(t))dX + \tfrac{1}{2}F''(X(t))\left[u\,dt + \sqrt{2D}\,dB\right]^2 + \dots$$
$$= F'(X(t))dX + DF''(X(t))\,dB^2 + O(dt^{3/2}) \quad \text{since } dB \sim dt^{1/2}$$

Therefore, to leading order in dt:

$$dF(X(t)) = F'(X(t))dX + DF''(X(t))dt$$
$$= \left[u F'(X(t)) + \underbrace{DF''(X(t))}_{!!!}\right]dt + F'(X(t))\sqrt{2D}\,dB(t)$$

!!!

[Note: If $F = F(X,t)$, we have:
$$dF = \frac{\partial F}{\partial X}dX + \frac{\partial F}{\partial t}dt + D\frac{\partial^2 F}{\partial X^2}dt$$
]

For a more general SDE

$$dX = f(X(t))dt + g(X(t))dB(t)$$

the „stochastic chain rule" for a function $F(X)$ is:

$$\boxed{dF(X(t)) = \left[f(X(t))F'(X(t)) + \tfrac{1}{2}g(X(t))^2F''(X(t))\right]dt + g(X(t))F'(X(t))dB(t)}$$

<u>Ito's Formula</u>

<u>Ex Matlab</u>: chainrule.m
$F = \sqrt{X}$, $dX = (\alpha - X)dt + \beta\sqrt{X}\,dB$, $X(0) = 1$
$\uparrow_2$ $\uparrow_1$

Financial mathematics ("an introduction")

Financial decisions are:
- How do I invest
- into what do I invest
- when do I sell etc.

Of course, risks are involved because we don't know from the past how the future markets will evolve.

|| What can we model using mathematics? ||

Some definitions:

- Any kind of investment (stocks, bonds, bank account) are called assets.
- The return of an asset ~~Si~~ with value $S(t)$ is defined as

$$\frac{S(t+\Delta t)-S(t)}{S(t)} = \frac{dS}{S}$$

[Why returns instead of $S(t)$ [prices]? Because returns are normalized by $S(t)$ → allow comparisons of stocks

- Mean return of an investment:

$$\mu = \frac{1}{N\Delta t}\sum \frac{S(t+\Delta t)-S(t)}{S(t)}$$

- Variance of an investment:

$$\mathrm{var}\left(\frac{dS}{S}\right) = \sigma^2\left(\frac{dS}{S}\right) = \frac{1}{(N-1)\Delta t}\sum_{n=1}^{N}\left(\frac{S(n\Delta t)-S((n-1)\Delta t)}{S((n-1)\Delta t)} - \mu\right)^2$$

- Long-term analysis of Standard & Poor's 500 index suggest

$$\left.\begin{array}{l}\mu \approx 7\% \quad (\Delta t = 1 \text{ year}) \\ \sigma \approx 15\%\end{array}\right\} \text{ for stocks}$$

A fundamental assumption of quantitative finance is that the return $\frac{dS}{S}$ ~~(...)~~ of a stock is a simple Brownian/Wiener process with drift μ and "diffusion" σ^2:

$$\boxed{\frac{dS}{S} = \mu\, dt + \sigma\, dB = \mu\, dt + \sigma \sqrt{dt}\, N(0,1)} \qquad (*)$$

Suppose $\sigma = 0 \Rightarrow \frac{dS}{S} = \mu dt$ and thus $\ln(S) = \mu t$ and so $S(t) = e^{\mu t}$, i.e. stock prices grow exponentially

What this model postulates is that the increments (returns) are affected by normal random "diffusion" (from whatever sources). Of course, this postulates has its origin in the central limit theorem - assuming that whether ~~or not~~ a stock increases or decreases is affected by many random influences that are uncorrelated, such that we would expect a Gaussian distribution of the stochastic part of $\frac{dS}{S}$. Correlations however become particularly important when a market is unstable, e.g. near a critical point; ~~...~~ precisely in this situation, the model $(*)$ breaks down... (at least when applied for returns over time scales that are not considerably larger than the correlation times).

One can show that the ~~...~~ model above ~~...~~ implies that $\ln(S)$ performs Brownian motion, i.e. a random walk.

Setting $\mu = 0$, we formally can write

$$\int \frac{dS}{S} = \int \sigma\, dB \quad \Rightarrow \quad \ln(S(t)) = \sigma B(t)$$

Thus, since $B(t)$ is a random walk, $\ln(S)$ is a random walk too. One says that S performs a lognormal random walk.

Note that $E\left[\frac{dS}{S}\right] = \mu\, dt$ and

$$\mathrm{var}\left[\frac{dS}{S}\right] = E\left[\left(\frac{dS}{S} - E\left(\frac{dS}{S}\right)\right)^2\right] = E\left[\sigma^2 dB^2\right] \sim \sigma^2 dt$$

as ~~the~~ we have seen before.

viz. $dX = \mu\, dt + \sqrt{2D}\, dB$
$\Rightarrow \mathrm{var}(dX) = 2D\, dt$

Black-Scholes equation

Assuming the above equation for stock prices. Investment therefore is risky because of random components. Clearly, there is a need for an investor to transfer his/her risk to others, who are less risk-averse, for a certain price (of course). This is quite similar to what insurances do. The financial instruments developed for this purpose are called derivatives. Of course, the trading of these derivatives implies that „risk" has a well-defined price. One such derivative is called an option.

Option: A contract that allows the holder to buy or sell an asset (stock etc.) at a fixed price in the future.
In contrast to a normal contract, the holder of an option doesn't need to make use of it („exercise" the option), only if it is convenient for him/her.

- call option: option to buy
- put " : " to sell
- expiration date: an option is only valid within a certain time window (American option), or only at a given date (European option)

Strategies using options:

- A stock holder might think that the market is overheated and is soon going to correct down. If he would sell his stocks, he might miss additional gains if the market doesn't (yet) go down. He therefore could buy a put option, allowing him to sell the stocks for a fixed price in a defined time-frame. I.e. he insures his stocks against a drop in price. To buy this insurance, the investor needs to pay upfront, for instance by selling some stocks. The put option therefore reduces a fraction of his potential gains.

- An investor holding a lot of stocks thinks that the stock isn't going to change in the near future. He therefore sells call options to make gains from the cost of the option. Since he already has the stocks, he can easily transfer them should the call options be exercised.

The big question is: How much to charge for an option?

Suppose the price of one call option is ~~$V_t = C(S_t, t)$~~ $V(t) = C(S(t), t)$ with C an unknown function.

fixed (could also write $C(S(t_0), t)$)

An investor sells one call option and buys Δ shares of S; then his total value changes over the interval $(t, t+dt)$ by

$$\cancel{(-V_{t+dt} + \Delta S_{t+dt}) - (-V_t + \Delta S_t)}$$

$$dA = \big(-V(t+dt) + \Delta S(t+dt)\big) - \big(-V(t) + \Delta S(t)\big) = -dV(t) + \Delta\, dS(t) \qquad (**)$$

Since $V(t) = C(S(t), t)$ and $\frac{dS}{S} = \mu dt + \sigma dB$ we can write using Ito's formula:

$$dV(t) = \frac{\partial C(S(t),t)}{\partial S} dS(t) + \frac{\partial C}{\partial t} dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt$$

Plugging into (**) we have

$$dA = \left(-\frac{\partial C}{\partial S} + \Delta\right) dS - \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt$$

If the investor would buy exactly $\Delta = +\frac{\partial C}{\partial S}$ stocks, the stochastic term vanishes. (i.e. hedging) Within the time period dt, this would be a risk-free investment (the eqn. is now an ODE).
Since every smart investor will try to do that, ~~so~~ the change of the portfolio is now only

$$dA = -\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt \qquad (***)$$

Efficient market theory now assumes that this risk-free (to order dt) investment cannot give more return than the short-term interest rate of a loan, r. Otherwise investors could borrow money at interest r, invest it as above and gain money in dt without risk (arbitrage opportunity).
~~Since~~ The value of the portfolio at time t is (as before)

$$-V(t) + \Delta S(t) \underset{\Delta = +\frac{\partial C}{\partial S}}{=} -C(S(t),t) + \frac{\partial C}{\partial S} S(t)$$

Over time dt, this amount of money ~~at~~ invested at interest r increases in value by

$$dA_r = \left(-C + \frac{\partial C}{\partial S}\right) dt \qquad (iv)$$

Postulating no arbitrage opportunities, we set $(***) = (iv)$ and obtain

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - S\frac{\partial C}{\partial S}\right)$$

or $$\boxed{\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0}$$ Black-Scholes equation (a PDE!)

To solve for C, we need boundary conditions; e.g.

$C(0,t) = 0 \quad \forall t$

$C(S,T) = \max(S(T) - K, 0)$: price of call option at maturity time T: if $S(T) > K$ (strike price), the option holder can buy the stock at price K at T and sell at current market price $S(T)$ with profit $S(T) - K$, so this should be price of the option.

If $S(T) < K$, the option is worthless $\rightarrow C = 0$.

Note these are not initial conditions but end conditions!

Under a change of variables

$\tau = T - t$

$u = Ce^{r\tau}$

$x = \ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau$

the BS eq. becomes the diffusion eq: $\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$

with initial conditions $u(x,0) = K\left(e^{\max(x,0)} - 1\right)$