

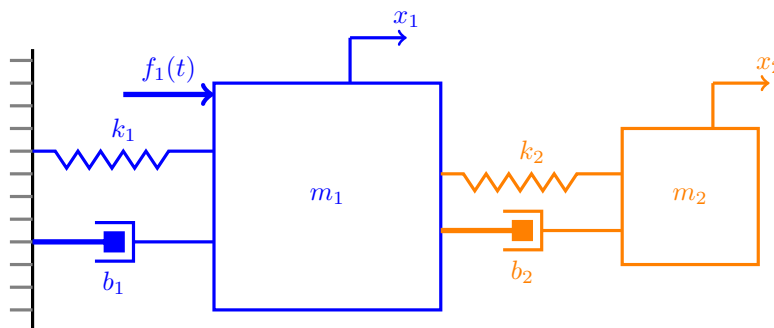
Coupled damped oscillators and the 18.031 Mascot

Tuned mass dampers

A tuned mass damper is a system of coupled damped oscillators in which one oscillator is regarded as primary and the second as a control or secondary oscillator. If tuned properly the maximum amplitude of the primary oscillator in response to a periodic driving force will be lowered and much of the energy will be absorbed by the secondary oscillator.

This is used for example in tall buildings to limit the swaying of the building in the wind. People are sensitive to this swaying, so by adding a tuned mass damper the building sways less and the damper, which no one can feel, vibrates instead. Another application is to stabilize laboratory tables supporting experiments that are sensitive to vibrations.

The 18.031 mascot is an example of such a system. The figure below represents an idealized version of it. The first mass m_1 is attached on one side to a wall by a spring and damper and on the other side it is attached to a second mass m_2 by another spring and damper. The spring and damping constants k_1, k_2, c_1, c_2 are indicated on the figure. A force $f_1(t)$ pushes on the first mass. The absolute positions of the masses are given by x_1 and x_2 , arranged so that the spring between them is relaxed when $x_1 = x_2$ and the spring connecting the first mass to the wall is relaxed when $x_1 = 0$.



We will regard first mass as the building, or the table – it's being shaken by some force, and we wish to control the amplitude of its resulting oscillation – so the system response of interest is x_1 . The second mass is presumably smaller, and the behavior of x_2 is of only secondary interest.

The system of differential equations

Newton's $F = ma$ and the usual assumptions about linear damping and spring force lead to the following differential equations governing the motion of the system.

$$\begin{aligned} m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1 - b_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1) &= f_1(t) \\ m_2 \ddot{x}_2 + b_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) &= 0 \end{aligned}$$

Let's rearrange these equations to put all the x_1 's on one side and all the x_2 's on the other:

$$\begin{aligned} (m_1 \ddot{x}_1 + b_1 \dot{x}_1 + k_1 x_1) + (b_2 \dot{x}_1 + k_2 x_1) &= f_1(t) + b_2 \dot{x}_2 + k_2 x_2 \\ m_2 \ddot{x}_2 + b_2 \dot{x}_2 + k_2 x_2 &= b_2 \dot{x}_1 + k_2 x_1. \end{aligned}$$

We can simplify the notation, and clarify the structure of these equations, by using operator notation. Define polynomials

$$\begin{aligned} P_1(s) &= m_1s^2 + b_1s + k_1 \\ P_2(s) &= m_2s^2 + b_2s + k_2 \\ Q_2(s) &= b_2s + k_2 \end{aligned}$$

Thus $P_1(s)$ is the characteristic polynomial of the first oscillator, $P_2(s)$ is the characteristic polynomial of the second oscillator, and $Q_2(s)$ is a first order polynomial reflecting the components connecting the two systems. Our system of equations becomes

$$\begin{aligned} (P_1(D) + Q_2(D))x_1 &= f_1(t) + Q_2(D)x_2 \\ P_2(D)x_2 &= Q_2(D)x_1 \end{aligned} \tag{1}$$

The system function

When we transform this system to the frequency domain we get

$$\begin{aligned} (P_1(s) + Q_2(s))X_1(s) &= F_1(s) + Q_2(s)X_2(s) \\ P_2(s)X_2(s) &= Q_2(s)X_1(s) \end{aligned} \tag{2}$$

(For simplicity, from now on we'll write P_1 , X_1 . etc instead of $P_1(s)$, $P_2(s)$, etc.)

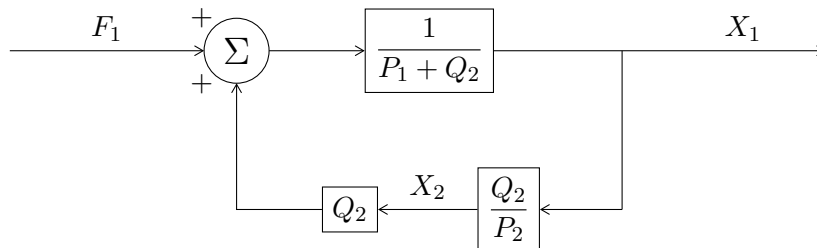
This is a pair of linear equations relating X_1 , X_2 , and F_1 . We're interested in X_1 , so let's begin by isolating X_1 on the left of the first equation:

$$X_1 = \frac{1}{P_1 + Q_2} (F_1 + Q_2X_2) \tag{3}$$

According to the second equation in (2),

$$X_2 = \frac{Q_2}{P_2} X_1. \tag{4}$$

Thus X_1 arises from summing F_1 with a multiple of X_2 ; and, X_2 is in turn a multiple of X_1 . We can interpret this as a feedback loop, and express it using the following block diagram.



Altogether the feedback branch, along the bottom, multiplies X_1 by Q_2^2/P_2 . Notice that the feedback is entirely determined by the parameters of the secondary oscillator.

It's easy enough to use (4) to eliminate X_2 from (3). Then using the fact that $P_2 - Q_2 = m_2s^2$, we find

$$X_1 = \frac{P_2}{P_1P_2 + m_2s^2Q_2} F_1$$

This has the expected form: in the frequency domain, the output signal X_1 is a certain function of s times the input signal; the transfer function for this system is

$$G_1(s) = \frac{P_2}{P_1 P_2 + m_2 s^2 Q_2}$$

Exercises: (a) Obtain this result using Black's formula.

(b) Draw the analogous block diagram assuming that x_2 is the system response of interest. What is the transfer function for it?

This formula contains several interesting pieces of information.

1. The zeros of the transfer function for this closed loop system occur precisely at the poles of the transfer function for the secondary oscillator – namely, at the roots of the characteristic polynomial $P_2(s) = m_2 s^2 + b_2 s + k_2$.

Suppose that the secondary oscillator is lightly damped; in fact, for simplicity, suppose that it is *undamped*. Then these roots will be $\pm i\omega_2$, where $\omega_2 = \sqrt{k_2/m_2}$ is the natural frequency of the secondary oscillator. Now drive the system with a sinusoidal force with angular frequency ω . If you now adjust the parameters m_2 and k_2 of the secondary oscillator so that $\omega_2 = \omega$, then the mass in the primary oscillator will be *stationary*: $G_1(i\omega) = 0$. This is ideal! An amazing feature of this result is that it can be made to function no matter what the secondary mass is. In reality there's always *some* damping, of course, but if it's small then the primary mass becomes *nearly* stationary when the driving frequency is near the natural frequency ω_2 .

Exercise: What's the downside to using very small mass for the secondary oscillator?

2. When $|s|$ is either very small or very large, the system behaves very much the way the main oscillator behaves – as if the secondary oscillator was just not there.

3. The transients of this system are of the form e^{rt} where r is a pole of $G_1(s)$ (assuming all these poles are simple). For almost all choices of system parameters, the roots of the numerator will differ from the roots of the denominator. In that case, the poles of $G_1(s)$ are precisely the roots of the denominator, $P_1(s)P_2(s) + m_2 s^2 Q_2(s)$. This is a fourth degree equation, so there are four independent transients. This makes sense, since there are four degrees of freedom in choosing initial conditions: you can set the position and the velocity of each of the two masses.