# Step and Delta Functions <br> 18.031 <br> Haynes Miller and Jeremy Orloff 

## 1 The unit step function

### 1.1 Definition

Let's start with the definition of the unit step function, $u(t)$ :

$$
u(t)= \begin{cases}0 & \text { for } t<0 \\ 1 & \text { for } t>0\end{cases}
$$

We do not define $u(t)$ at $t=0$. Rather, at $t=0$ we think of it as in transition between 0 and 1 .
It is called the unit step function because it takes a unit step at $t=0$. It is sometimes called the Heaviside function. The graph of $u(t)$ is simple.


We will use $u(t)$ as an idealized model of a natural system that goes from 0 to 1 very quickly. In reality it will make a smooth transition, such as the following.


Figure 1. $u(t)$ is an idealized version of this curve
But, if the transition happens on a time scale much smaller than the time scale of the phenomenon we care about then the function $u(t)$ is a good approximation. It is also much easier to deal with mathematically.
One of our main uses for $u(t)$ will be as a switch. It is clear that multiplying a function $f(t)$ by $u(t)$ gives

$$
u(t) f(t)= \begin{cases}0 & \text { for } t<0 \\ f(t) & \text { for } t>0\end{cases}
$$

We say the effect of multiplying by $u(t)$ is that for $t<0$ the function $f(t)$ is switched off and for $t>0$ it is switched on.

### 1.2 Integrals of $u^{\prime}(t)$

From calculus we know that

$$
\int u^{\prime}(t) d t=u(t)+c \quad \text { and } \quad \int_{a}^{b} u^{\prime}(t) d t=u(b)-u(a)
$$

For example:

$$
\begin{aligned}
& \int_{-2}^{5} u^{\prime}(t) d t=u(5)-u(-2)=1 \\
& \int_{1}^{3} u^{\prime}(t) d t=u(3)-u(1)=0 \\
& \int_{-5}^{-3} u^{\prime}(t) d t=u(-3)-u(-5)=0
\end{aligned}
$$

In fact, the following rule for the integral of $u^{\prime}(t)$ over any interval is obvious

$$
\int_{a}^{b} u^{\prime}(t)= \begin{cases}1 & \text { if } 0 \text { is inside the interval }(a, b)  \tag{1}\\ 0 & \text { if } 0 \text { is outside the interval }[a, b]\end{cases}
$$

Note: If one of the limits is 0 we throw up our hands and refuse to do the integration. Let $0^{-}$be infinitesimally to the left of 0 and $0^{+}$infinitesimally to the right of 0 . That is,

$$
0^{-}<0<0^{+} .
$$

For a function, $f\left(0^{-}\right)$is defined as the left hand limit at 0 or equivalently the limit from below at 0 , provided this limit exists. Likewise $f\left(0^{+}\right)$is the right hand limit or the limit from above.

$$
f\left(0^{-}\right)=\lim _{t \uparrow 0} f(t) \quad f\left(0^{+}\right)=\lim _{t \downarrow 0} f(t)
$$

Here are some examples of integrals of $u^{\prime}$ that involve $0^{-}$and $0^{+}$:

$$
\begin{aligned}
& \int_{-\infty}^{0^{+}} u^{\prime}(t) d t=1 \quad\left(\text { because }-\infty<0<0^{+}\right) \\
& \int_{-\infty}^{0^{-}} u^{\prime}(t) d t=0 \quad\left(\text { because }-\infty<0^{-}<0\right) \\
& \int_{0^{-}}^{0^{+}} u^{\prime}(t) d t=1 \quad\left(\text { because } 0^{-}<0<0^{+}\right)
\end{aligned}
$$

### 1.3 Preview of generalized functions and derivatives

Of course $u(t)$ is not a continuous function, so in the 18.01 sense its derivative at $t=0$ does not exist. Nonetheless we saw that we could make sense of the integrals of $u^{\prime}(t)$. So rather than throw it away we call $u^{\prime}(t)$ the generalized derivative of $u(t)$. You can't do everything with $u^{\prime}(t)$ you can do with an ordinary function, but it can go anywhere we have an input function in 18.03. In the next section we will look in more detail at $u^{\prime}(t)$-and call it $\delta(t)$. For now we'll content ourselves with computing the Laplace transform of $u$ and $u^{\prime}$.

### 1.4 The Laplace transform of $u(t)$ and $u^{\prime}(t)$

This is easy since $u(t)$ is identical to the constant function 1 on the interval $(0, \infty)$ of the Laplace transform. Therefore

$$
\mathcal{L}(u(t))=1 / s .
$$

(We could also compute this directly from the definition $\mathcal{L}(u)=\int_{0^{-}}^{\infty} u(t) e^{-s t} d t$.)
For $u^{\prime}$, we use the formula $\mathcal{L}\left(u^{\prime}\right)=s \mathcal{L}(u)-u\left(0^{-}\right)$and the fact that $u\left(0^{-}\right)=0$ to get

$$
\mathcal{L}\left(u^{\prime}\right)=s \cdot \frac{1}{s}=1 .
$$

### 1.5 The unit step response

Suppose we have an LTI system with system function $H(s)$. The unit step response of this system is defined as its response to input $u(t)$ with rest initial conditions.
Theorem. The Laplace transform of the unit step response is $H(s) \frac{1}{s}$.
Proof. This is a triviality since in the frequency domain: output $=$ transfer function $\times$ input.
Example 1. Consider the system $\dot{x}+2 x=f(t)$, with input $f$ and response $x$. Find the unit step response.
answer: We have $f(t)=u(t)$ and rest initial conditions. The system function is $1 /(s+2)$, so by the theorem, the unit step response written in terms of frequency is given by

$$
X(s)=\frac{1}{s(s+2)}
$$

The partial fractions decomposition is $X(s)=\frac{1}{2}\left(\frac{1}{s}-\frac{1}{s+2}\right)$, so in the time domain the unit step response is $x(t)=\frac{1}{2}-\frac{1}{2} e^{-2 t} \quad$ for $t>0$. (Of course $x(t)=0$ for $t<0$.)

Example 2. In the previous example, find the long-term behavior of the unit step response in two ways.
answer: Method 1: Compute the limit directly.

$$
\lim _{t \rightarrow \infty} x(t)=\lim _{t \rightarrow \infty} \frac{1}{2}-\frac{1}{2} e^{-2 t}=\frac{1}{2}
$$

Method 2: Use the final value theorem. (If you haven't covered that in class just skip this method -or go back and read about the final value theorem in the reading on Laplace transform.) We have $s X(s)=1 /(s+2)$. Since all its poles are negative, we can apply the final value theorem:

$$
\lim _{t \rightarrow \infty} x(t)=\lim _{s \rightarrow 0} s X(s)=\frac{1}{2}
$$

We see that both methods agree!

## 2 The unit impulse

In this section we will learn about the unit impulse function $\delta(t)$. We will use it as input to LTI systems. At first the systems will be simple enough to find the postinitial conditions directly and use them to solve the equations for the response. For more complicated systems we will use the Laplace transform to solve the equation without first determining the post-initial conditions.

### 2.1 The mathematics of the delta function

Let's delve a little deeper into $u^{\prime}(t)$. It's clear $u^{\prime}(t)=0$ if $t \neq 0$. At $t=0$ the curve is vertical so the slope is infinite, i.e. $u^{\prime}(0)=\infty$. (If you think of $u(t)$ as an idealized version of the curve in Figure 1, then we would say the derivative near 0 gets very large.) We define

$$
\delta(t)=u^{\prime}(t)
$$

and call it the delta function or the Dirac delta function or the unit impulse function. We have seen the following properties of $\delta(t)$.

1. $\delta(t)= \begin{cases}0 & \text { if } t \neq 0 \\ \infty & \text { if } t=0 .\end{cases}$
2. $\int \delta(t) d t=u(t) \quad$ and $\quad \int_{-\infty}^{\infty} \delta(t) d t=1$.

Based on property 1, we 'graph' $\delta(t)$ as an infinite spike at the origin. The integrals show that the 'area' under this graph equals 1 and it is all concentrated at the origin.


We also show $\delta(t-a)$ which is just $\delta(t)$ shifted to the right.

### 2.2 The non-idealized delta function

Just like the unit step function, the $\delta$ function is really an idealized view of nature. In reality, a delta function is nearly a spike near 0 which goes up and down on a time interval much smaller than the scale we are working on. The integral, i.e. area under the curve, is always 1. It's graph might actually look something like


Figure 2. Non-idealized delta function; area under the graph $=1$.
The total amount input is still the integral (see Section 2.4 below), or, in geometric terms, the area under the graph. For a unit impulse we assume the area is 1.

### 2.3 Delta functions are your friend

### 2.3.1 Integrals with $\delta(t)$

Recall how painful integration could be. In contrast, integrals with delta functions are always easy and involve no techniques of integration.

Suppose we scale $\delta(t)$ : the integrals are just scaled.

$$
\int_{-5}^{5} 3 \delta(t) d t=3, \quad \int_{-5}^{-3} 3 \delta(t) d t=0, \quad \int_{0^{-}}^{0^{+}} 3 \delta(t) d t=3, \quad \int_{0^{+}}^{\infty} 3 \delta(t) d t=0
$$

The integral $\int_{a}^{b} f(t) \delta(t) d t$ is also easy. If $f(t)$ is continuous at $t=0$ then

$$
\int_{a}^{b} f(t) \delta(t) d t= \begin{cases}f(0) & \text { if }(a, b) \text { contains } 0 \\ 0 & \text { if }[a, b] \text { does not contain } 0\end{cases}
$$

That is, integrating against $\delta(t)$ just amounts to evaluating $f(t)$ at $t=0$.

Note 1. If one of the endpoints $a$ or $b$ is 0 , the integral cannot be evaluated, so we just throw up our hands and refuse to do it.
Note 2. Technicality: We must have $f(t)$ continuous at $t=0$.

### 2.3.2 Justification of the formula for $\int f(t) \delta(t) d t$

We should start by admitting that in formal mathematics this is simply given as the definition of $\delta(t)$, so our arguments will just go to show that it is a reasonable definition. We'll do this in three ways.

Quick reason: $\delta(t)$ is 0 everywhere except $t=0$, So $f(t) \delta(t)$ is 0 for all $t \neq 0$ and at $t=0$ it just scales the delta function by $f(0)$. That is, $f(t) \delta(t)=f(0) \delta(t)$.
Reason 1. Since we can interpret the integral as area, we need to show that the 'area' under $f(t) \delta(t)$ is $f(0)$. Figure 2 (above) shows a tall, thin curve near $t=0$ which approximates $\delta(t)$. Since $f(t)$ is continuous we know that $f(t) \approx f(0)$ near $t=0$. Thus, $f(t) \delta(t)$ is approximated by the graph in the figure scaled by $f(0)$. Finally, since the area under the curve in Figure 2 is one, if we scale it by $f(0)$ it will have area equal to $f(0)$. As the graph in Figure 2 gets narrower and taller it goes to that of $\delta(t)$. As this happens, the approximation we just made will become exact, i.e. as we wanted to show, the area under the $f(t) \delta(t)=f(0)$.

Reason 2. This is a direct argument using integration by parts. First, since $\delta(t)=0$ for $t \neq 0$ the integral $\int_{a}^{b} f(t) \delta(t) d t$ must be zero for any interval $[a, b]$ not containing 0 . Next, suppose $a<0<b$, then we get

$$
\begin{aligned}
\int_{a}^{b} f(t) \delta(t) d t & =\int_{a}^{b} f(t) u^{\prime}(t) d t \quad\left(\text { since } \delta=u^{\prime}\right) \\
& =\left.f(t) u(t)\right|_{a} ^{b}-\int_{a}^{b} f^{\prime}(t) u(t) d t \quad \text { (integration by parts) }
\end{aligned}
$$

Now, since $u(b)=1, u(a)=0$ and $u(t)=0$ for $t<0$ this becomes

$$
\begin{aligned}
& =f(b)-\int_{0}^{b} f^{\prime}(t) d t \\
& =f(b)-\left.f(t)\right|_{0} ^{b} \\
& =f(b)-f(b)+f(0) \\
& =f(0)
\end{aligned}
$$

Comparing the first and last expressions in this long sequence of steps, we've shown the result.

### 2.3.3 Shifting by a

If we shift by $a$ we have, $\int_{-\infty}^{\infty} f(t) \delta(t-a)=f(a)$. More generally:

$$
\int_{c}^{d} f(t) \delta(t-a) d t= \begin{cases}f(a) & \text { if }(c, d) \text { contains a } \\ 0 & \text { if }[c, d] \text { does not contain a. }\end{cases}
$$

Example 3. (Practice with $\delta$.) Quickly cover up the answers on the left and try to evaluate each of the integrals on the right.

$$
\begin{aligned}
& \int_{-1}^{3} \delta(t) 2 e^{4 t^{2}} d t \quad=2, \quad\left(\text { evaluate } 2 e^{4 t^{2}} \text { at } t=0\right) \\
& \int_{1}^{3} \delta(t) 2 e^{4 t^{2}} d t \quad=0, \quad(0 \text { is not in }[1,3]) \\
& \int_{0^{-}}^{3} \delta(t) 2 e^{4 t^{2}} d t \quad=2, \quad\left(\text { evaluate } 2 e^{4 t^{2}} \text { at } t=0\right) \\
& \int_{0^{-}}^{\infty} \delta(t) 2 e^{-\tan ^{2}\left(t^{3}\right)} d t \quad=2, \quad\left(\text { evaluate } 2 e^{-\tan ^{2}\left(t^{3}\right)} \text { at } t=0\right) \\
& \int_{-1}^{3} \delta(t-2) 2 e^{4 t^{2}} d t \quad=2 e^{16}, \quad\left(\text { evaluate } 2 e^{2 e^{4 t^{2}}} \text { at } t=2\right) \\
& \int_{3}^{5} \delta(t-2) 2 e^{4 t^{2}} d t \quad=0, \quad(2 \text { is not in }[3,5]) \\
& \int_{0^{-}}^{3} \delta(t-2) 2 e^{4 t^{2}} d t \quad=2 e^{16} \quad\left(\text { evaluate } 2 e^{2 e^{4 t^{2}}} \text { at } t=2\right) \text {, } \\
& \int_{0^{-}}^{\infty} \delta(t-2) 2 e^{-\tan ^{2}\left(t^{3}\right)} d t=2 e^{-\tan ^{2}(8)} \quad \text { (evaluate } 2 e^{-\tan ^{2}\left(t^{3}\right)} \text { at } t=2 \text { ). }
\end{aligned}
$$

### 2.4 The physical interpretation of delta functions as a unit impulse

In general, we will be using $\delta$ functions as the input to LTI systems. So, in this subsection we want to explore what this means. Our goal is to understand what is meant by an impulse and to see that $\delta(t)$ can be thought of as an (idealized) unit impulse.
Example 4. Consider the rate equation $\dot{x}+k x=f(t)$. To be specific, assume $x$ is in units of kilograms and $t$ is in minutes. This is a rate equation and the derivative $\dot{x}$ and the input $f(t)$ are rates, in units of $\mathrm{kg} / \mathrm{min}$. We then have that the total amount of kg input from time $0^{-}$to time $t$ is $\int_{0^{-}}^{t} f(\tau) d \tau$.
Consider the following possible inputs $f(t)$, shown graphically as box functions.


Look at the input function $f_{1}(t)$ in the leftmost figure. It is only nonzero in the interval $[0,1 / 2]$ during which time it inputs at a constant rate of $2 \mathrm{~kg} / \mathrm{min}$. The total amount input over that time is

$$
\int_{0}^{1 / 2} f_{1}(t) d t=1 \mathrm{~kg}
$$

The function $f_{2}$ has a higher rate, but acts for a shorter time. The total amount it inputs over time is also 1 kg . The function $f_{3}$ is similar: it acts for even a shorter time, but inputs a total of 1 kg .
If $x(0)=x_{0} \mathrm{~kg}$, then over the interval $[0,1 / 2]$ some of what is added by $f_{1}(t)$ will decay away and we'll end with something less than $x_{0}+1 \mathrm{~kg}$. But, the shorter the interval the less time there is to decay and the amount at the end will be closer to $x_{0}+1$. If we continue to shorten the time interval in which we input a total of 1 kg then in the limiting case we will dump 1 kg in all at once. In this case there will be no time for decay and the amount will jump instantaneously from $x_{0}$ to $x_{0}+1$, after which it will start decaying. This instantaneous input is called an impulse; an instantaneous input of one unit is called a unit impulse. In a first order system an impulse results in an instantaneous jump in the amount of $x$.
Note that as the length of the time interval goes to 0 , the rate $f(t)$ must go to infinity. But all along the total kg input remains constant. That is the integral $\int_{-\infty}^{\infty} f(t) d t=1$.
Claim. Let $u_{h}(t)$ be the box function of width $h$ and height $1 / h$. Then the integral $\int_{-\infty}^{\infty} u_{h}(t) d t=1$ and

$$
\lim _{h \rightarrow 0} u_{h}(t)=\delta(t)
$$

That is, as the boxes get narrower and taller they become the $\delta$ function.
Proof. We saw above that $\delta(t)$ was described by two properties

1. $\delta(t)= \begin{cases}0 & \text { if } t \neq 0 \\ \infty & \text { if } t=0 .\end{cases}$
2. $\int \delta(t) d t=u(t), \quad \int_{-\infty}^{\infty} \delta(t) d t=1$.

The picture below illustrates that $\lim _{h \rightarrow 0} u_{h}(t)$ satisfies property 1 . Because all the integrals of $u_{h}(t)=1$ the second property is also true of the limit. Because the limit satisfies both properties it must equal $\delta(t)$.


A sequence of box functions $u_{h}(t)$ limiting to $\delta(t)$.
Summary. Here's a summary of what we've done in this subsection.

1. If $f(t)$ is an input rate. The total amount input over $[a, b]$ is $\int_{a}^{b} f(t) d t$.
2. A unit impulse adds a total of 1 unit in one instant.
3. If the impulse is at $t=t_{0}$ then all the input happens at $t=t_{0}$.
4. We can visualize an impulse as the limit of a sequence of boxes as they get narrower and taller. (Also, look back at the non-idealized delta function in Figure 2: an impulse is the limit of any spike function as it gets narrower and taller.)
5. A unit impulse is modeled by $\delta(t)$.

### 2.5 Solving DES: pre and post-initial conditions.

The main lesson in this section is that for an $n$th order equation a delta function input causes an instantaneous jump in the $(n-1)$ st derivative of the output. Once we deal with that we can use our standard techniques to solve the DE.

Because an impulse causes an instantaneous jump in some value, we have to consider the conditions just before and just after the impulse. Assume the impulse occurs at $t=0$ then

At $t=0^{-}$: the conditions are pre-initial conditions
At $t=0^{+}$: the conditions are post-initial conditions

### 2.5.1 Impulses as input to first order systems

Example 5. Solve $\dot{x}+k x=\delta(t)$ with rest initial conditions.
answer: This is a first order exponential decay system. The unit impulse at $t=0$ causes an instantaneous jump of 1 in the value of $x$. Since the pre-initial condition is $x\left(0^{-}\right)=0$, the post-IC must be $x\left(0^{+}\right)=1$. Since the input function $\delta(t)$ is 0 for $t>0$ we have a homogenous initial value problem on this interval:

$$
\dot{x}+k x=0, \text { for } t>0, \quad x\left(0^{+}\right)=1
$$

The solution for $t>0$ is $x(t)=e^{-k t}$ (you can check $x\left(0^{+}\right)=1$ ). The full solution is

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ e^{-k t} & \text { for } t>0\end{cases}
$$



Response from rest to input $=\delta(t)$.

### 2.5.2 Impulses as imput to second order systems

Now let's consider the second order system

$$
\begin{equation*}
m \ddot{x}+b \dot{x}+k x=f(t) \tag{2}
\end{equation*}
$$

with input $f(t)$ and output $x(t)$. To be specific, we'll think of this as a spring-massdamper system, with $x$ in meters, $t$ in seconds, and $m$ in kg .
We need to think about the units on $f(t)$. It's clear enough that they are in Newtons, but what are the units of the total input $\int_{a}^{b} f(t) d t$ ? Newtons can be written as

$$
\text { Newton }=\frac{\mathrm{kg} \cdot \mathrm{~m} / \mathrm{sec}}{\mathrm{sec}}=\frac{\text { momentum }}{\text { time }} .
$$

That is, force changes momemtum over time. We see that the total input has units of momentum.

Following this idea we see that a unit impulse to this second order system is a sudden blow, i.e. a large force acting with a short duration, that causes the momentum to jump by one unit.
Example 6. Suppose a unit impulse is applied to the system in Equation 2. If the system is at rest before time 0 , find the pre- and post-initial conditions.
answer: Since the system is initially at rest the pre-initial conditions are

$$
x\left(0^{-}\right)=0 \quad \text { and } \quad \dot{x}\left(0^{-}\right)=0 .
$$

Since, for this system, the impulse causes a one unit jump in momentum at $t=0$ we have, at $t=0^{+}$, the momentum $m \dot{x}\left(0^{+}\right)=1$, i.e. the post-initial conditions

$$
x\left(0^{+}\right)=0 \quad \text { and } \quad \dot{x}\left(0^{+}\right)=1 / m .
$$

Example 7. Assume rest initial conditions and solve the equation

$$
2 \ddot{x}+7 \dot{x}+3 x=\delta(t) .
$$

answer: Following the previous example the post-initial conditions are $x\left(0^{+}\right)=0$ and $\dot{x}\left(0^{+}\right)=1 / 2$. For $t>0$, the input $\delta(t)=0$, so we have a homogeneous equation

$$
2 \ddot{x}+7 \dot{x}+3 x=0, \quad x\left(0^{+}\right)=0, \dot{x}\left(0^{+}\right)=1 / 2, \text { for } t>0 .
$$

The characteristic roots are $-1 / 2$ and -3 , so for $t>0$ we have

$$
x(t)=c_{1} e^{-t / 2}+c_{2} e^{-3 t}
$$

We can find $c_{1}$ and $c_{2}$ to match the post-initial conditions:

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ \frac{1}{5} e^{-t / 2}-\frac{1}{5} e^{-3 t} & \text { for } t>0\end{cases}
$$

Example 8. Solve $4 \ddot{x}+x=\delta(t)$ with rest IC.
answer: The pre-inital conditions are 0 , so the post-initial conditions are

$$
x\left(0^{+}\right)=0, \quad 4 \dot{x}\left(0^{+}\right)=1 .
$$

For $t>0$ the differential equation becomes $4 \ddot{x}+x=0$. We know the solution to this:

$$
x(t)=c_{1} \cos (t / 2)+c_{2} \sin (t / 2) \quad \text { for } t>0 .
$$

We find $c_{1}$ and $c_{2}$ to match the post-initial conditions: $c_{1}=0, c_{2}=1 / 2$. Therefore the complete solution is

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ \frac{1}{2} \sin (t / 2) & \text { for } t>0\end{cases}
$$

Physical explanation. At $t=0$ an impulse kicks the simple harmonic oscillator into motion. After that input is 0 and the system is in simple harmonic motion. The jump in momentum corresponds to the corner in graph at 0 .


Example 9. Solve $4 \ddot{x}+x=\delta(t-a)$ with rest IC.
answer: This is an LTI system so shifting the input from the previous example $a$ units to the right just shifts the response in the same way.


Example 10. (Resonance) Solve the equation $\ddot{x}+x=f(t)$ with rest IC, where the input $f(t)$ is an impulse every $2 \pi$ seconds of magnitude 3 in the positive direction.
answer: We have $f(t)=3 \delta(t)+3 \delta(t-2 \pi)+3 \delta(t-4 \pi)+\ldots$. We can solve by solving the DE individually for each input:

$$
\ddot{x}_{n}+x_{n}=3 \delta(t-2 n \pi)
$$

and using superpostion. The individual equations are exactly like the previous example. We get that the solution to $\ddot{x}_{n}+x_{n}=3 \delta(t-2 n \pi)$ is

$$
x_{n}(t)=\left\{\begin{array}{cl}
0 & \text { for } t<2 n \pi \\
3 \sin (t-2 n \pi)=3 \sin (t) & \text { for } t>2 n \pi
\end{array}\right.
$$

Now when we superposition these solutions we see that every $2 \pi$ seconds we add another copy of $3 \sin (t)$ to the output. We call this resonance -the blows come at the natural frequency (every $2 \pi$ seconds) of the system.

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ 3 \sin (t) & \text { for } 0<t<2 \pi \\ 6 \sin (t) & \text { for } 2 \pi<t<4 \pi \\ 9 \sin (t) & \text { for } 4 \pi<t<6 \pi \\ & \cdots\end{cases}
$$

### 2.5.3 Impulses as input to third order systems

Example 11. Assume rest initial conditions and solve the equation

$$
4(D-1)(D-2)(D-3) x=4 x^{\prime \prime \prime}-24 x^{\prime \prime}+44 x^{\prime}-24 x=5 \delta(t)
$$

(We give the differential operator in factored form so we can find the characteristic roots easily.)
answer: This is a third order DE , so the input $5 \delta(t)$ causes a jump in $x^{\prime \prime}(t)$ at $t=0$. Since the leading coefficient is 4 the jump has magnitude $5 / 4$. The pre-initial conditions are all zero, so after the jump the post-initial conditions are

$$
x\left(0^{+}\right)=0, \quad x^{\prime}\left(0^{+}\right)=0, \quad x^{\prime \prime}\left(0^{+}\right)=5 / 4 .
$$

For $t>0$, the input $\delta(t)=0$, so we have a homogeneous initial value problem

$$
4(D-1)(D-2)(D-3) x=0, \quad x\left(0^{+}\right), x^{\prime}\left(0^{+}\right)=0, x^{\prime \prime}\left(0^{+}\right)=5 / 4, \text { for } t>0
$$

The characteristic roots are 1,2 and 3 , so for $t>0$ we have

$$
x(t)=c_{1} e^{t}+c_{2} e^{2 t}+c_{3} \mathrm{e}^{3 t} .
$$

It is easy to use the initial conditions to find the coefficients:

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ \frac{5}{8} e^{t}-\frac{5}{4} e^{2 t}+\frac{5}{8} e^{3 t} & \text { for } t>0\end{cases}
$$

### 2.6 Box vs. delta as input

In this section we will compare box functions and delta functions as input. You will see that the delta function is much easier to work with!

Example 12. (Box vs. delta.) Let's compare box $u_{h}(t)$ input with unit impulse $(\delta(t))$ input by solving: $\dot{x}+k x=u_{h} \quad$ with rest IC.
(Physical reasoning:) This models radioactive dumping. $u_{h}$ is the rate matter is added over time and as we have seen the total amount added is $\int_{0}^{h} u_{h}=1$.
In the figure below the top row of graphs show the input $u_{h}$ for various values of $h$. The corresponding responses are shown in the second row of graphs. The total amount input is one, so, since there is decay, at the end of the interval we have $x(h)<1$. After time $t=h$ there is no more input and the response shows exponential decay.
As $h$ goes to 0 the input becomes the unit impulse $\delta(t)$. This is shown in the last graph. Since the input is dumped in all at once the graph jumps from 0 to 1 at $t=0$. After $t=0$ the graph is that of exponential decay.


Top: a sequence of box function inputs limiting to $\delta(t)$.
Bottom: response to the sequence of box functions limiting to response to $\delta(t)$.

For completeness we give the exact solution to the IVP $\dot{x}+k x=u_{h}$ with rest IC.

$$
x= \begin{cases}\frac{1}{h k}\left(1-e^{-k t}\right) & \text { for } 0<t<h \\ \frac{1}{h k}\left(e^{k h}-1\right) e^{-k t} & \text { for } h<t\end{cases}
$$

Just as expected, as $h \rightarrow 0$ the input becomes $\delta$ and the output becomes $x=e^{-k t}$ (i.e. $\lim _{h \rightarrow 0} \frac{e^{k h}-1}{h k}=1$ )

## 3 Generalized derivatives

So far we have only one generalized derivative: $\dot{u}(t)=\delta(t)$. In this section we will learn to compute them for any function with jump discontinuities.
Definition. We say a function $f(t)$ has a jump discontinuity at $t=t_{0}$ if its graph is continuous on both the left and right, and there is a jump at $t_{0}$.

Formally this means that both left and right limits $\lim _{t \uparrow t_{0}^{-}} f(t)$ and $\lim _{t \downarrow t_{0}^{+}} f(t)$ exist, but are different. The jump at $t_{0}$ is defined as the difference

$$
\lim _{t \rightarrow t_{0}^{+}} f(t)-\lim _{t \rightarrow t_{0}^{-}} f(t)
$$

Example 13. The graph of the function $f(t)$ is shown below. It has jump discontinuites at $-2,0$ and 2 . The jumps are respectively $2,-2$ and 3 . The graph also has a corner at -1 . That is, the graph is continuous at $t=-1$, but the derivative has a jump there.


Notes. 1. Not all discontinuties result in jumps. At $t=1$ the jump between the left and right limits is 0 . You could say the function jumps from -1.5 to 0 and back to -1.5 for a net jump of 0 .
2. The value of $f(2)$ (represented by a dot on the graph) did not play a role in the value of the jump at $t=2$. The jump is the size of gap between the left and right
branches of the curve. You could say the function jumps from 0 to 1.5 to 3 for a net jump of 3 .
3. At $t=0$ the jump is negative because the right branch of the graph is below the left branch.

The generalized derivative of a function that is smooth except for some jump discontinuties and corners is the regular derivative away from the jumps and corners, delta functions at the jumps and undefined at the corners. The coefficient on the delta function is the size of the jump.

Reason. Just as with the unit step function the graph has 'infinite' slope at a jump and the integral of the derivative should give the original function. This is exactly what $\delta$ functions do at jumps.

Example 14. The function in the previous example is

$$
f(t)= \begin{cases}-2 & \text { for } t<-2 \\ t+2 & \text { for }-2<t<-1 \\ -t & \text { for }-1<t<0 \\ x^{2} / 2-2 & \text { for } 0<t<1 \text { and } 1<t<2 \\ 3-3(x-2)^{2} & \text { for } 2<t\end{cases}
$$

Find its generalized derivative.
answer: We just take the regular derivative and add delta functions at the jump discontinuities. Note that the corner when $t=-1$ becomes a jump in the derivative.

$$
f^{\prime}(t)=2 \delta(t+2)-2 \delta(t)+3 \delta(t-3)+ \begin{cases}0 & \text { for } t<-2 \\ 1 & \text { for }-2<t<-1 \\ -1 & \text { for }-1<t<0 \\ x & \text { for } 0<t<1 \text { and } 1<t<2 \\ -6(x-2) & \text { for } 2<t\end{cases}
$$

Example 15. Derivative of a square wave
The graphs below are of a function $\mathrm{sq}(t)$ (called a square wave) and its derivative. The function alternates every $\pi$ seconds between $\pm 1$. The derivative $\mathrm{sq}^{\prime}(t)$ is clearly 0 everywhere except at the jumps. A jump of +2 gives a (generalized) derivative of $2 \delta$ and a jump of -2 gives a (generalized) derivative of $-2 \delta$. Thus we have
$\mathrm{sq}^{\prime}(t)=\ldots+2 \delta(t+2 \pi)-2 \delta(t+\pi)+2 \delta(t)-2 \delta(t-\pi)+2 \delta(t-2 \pi)-2 \delta(t-3 \pi)+\ldots$


Graph of $\mathrm{sq}(t)=$ square wave


Graph of $\mathrm{sq}^{\prime}(t)=$ impulse train

Note that we put the weight of each delta function next to it. We use the convention that $-2 \delta(t)$ is represented by a downward arrow with the weight 2 next to it. That is, the sign is represented by the direction of the arrow, so the weight is positive.

### 3.1 Using the generalized derivative to check solutions

In this section we will check the answers to a few of our previous examples by plugging them into the original DE . This should give you a better feel for why a delta function as input causes a jump in the $(n-1)$ st derivative.
Example 16. (Check the solution in Example 5)
The DE $\quad \dot{x}+k x=\delta(t)$ has solution $x(t)=\left\{\begin{array}{ll}0 & \text { for } t<0 \\ \mathrm{e}^{-k t} & \text { for } t>0 .\end{array}\right.$.
This has a jump of 1 at $t=0$, so $\dot{x}(t)$ is a generalized derivative:

$$
\dot{x}(t)=\delta(t)+ \begin{cases}0 & \text { for } t<0 \\ -k \mathrm{e}^{-k t} & \text { for } t>0\end{cases}
$$

It is now easy to verify that $\dot{x}+k x=\delta(t)$.
Notice that the jump in $x$ yielded a delta function in $\dot{x}$.
Example 17. (Check Example 7) Here the DE was $2 \ddot{x}+7 \dot{x}+3 x=\delta(t)$ and the solution was

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ \frac{1}{5} e^{-t / 2}-\frac{1}{5} e^{-3 t} & \text { for } t>0\end{cases}
$$

$x(t)$ has no jump at $t=0$, so its derivative is

$$
\dot{x}(t)= \begin{cases}0 & \text { for } t<0 \\ -\frac{1}{10} e^{-t / 2}+\frac{3}{5} e^{-3 t} & \text { for } t>0\end{cases}
$$

Since $\dot{x}(t)$ has a jump of $1 / 2$ at $t=0$, we will get a $\delta$ function in $\ddot{x}(t)$ :

$$
\ddot{x}(t)=\frac{1}{2} \delta(t)+ \begin{cases}0 & \text { for } t<0 \\ \frac{1}{20} e^{-t / 2}-\frac{9}{5} e^{-3 t} & \text { for } t>0\end{cases}
$$

It is now easy to check that $2 \ddot{x}+7 \dot{x}+3 x=\delta(t)$.
Notice that the jump in $\dot{x}(t)$ yielded a delta function in $\ddot{x}(t)$-this is an algebraic explanation of why a delta function input to an $n$th order DE causes a jump in the $n-1$ derivative.

Example 18. (Check Example 11) We will do this check more quickly than the previous two. Also, we will leave out the case $t<0$ since it is always 0 . As we do the computation notice that $x\left(0^{+}\right)=x^{\prime}\left(0^{+}\right)=0$, so there is no jump until $x^{\prime \prime}\left(0^{+}\right)=5 / 4$.

The DE is $4 x^{\prime \prime \prime}-24 x^{\prime \prime}+44 x^{\prime}-24 x=5 \delta(t)$.

$$
\begin{aligned}
-24 x & =-24\left(\frac{5}{8} \mathrm{e}^{t}-\frac{5}{4} \mathrm{e}^{2 t}+\frac{5}{8} \mathrm{e}^{3 t}\right) \\
44 x^{\prime} & =44\left(\frac{5}{8} \mathrm{e}^{t}-\frac{5}{2} \mathrm{e}^{2 t}+\frac{15}{8} \mathrm{e}^{3 t}\right) \\
-24 x^{\prime \prime} & =-24\left(\frac{5}{8} \mathrm{e}^{t}-5 \mathrm{e}^{2 t}+\frac{45}{8} \mathrm{e}^{3 t}\right) \\
4 x^{\prime \prime \prime} & =4\left(\frac{5}{8} \mathrm{e}^{t}-10 \mathrm{e}^{2 t}+\frac{135}{8} \mathrm{e}^{3 t}+\frac{5}{4} \delta(t)\right)
\end{aligned}
$$

Adding this up verifies that $x(t)$ is a solution to the DE.

### 3.2 Generalized functions: regular and singular parts

We finish this section by introducing a bit of useful vocabulary. When we computed the generalized derivative of a function we had to do two things:

1. Where it made sense we took the regular derivative and got a regular function
2. At jumps we took the generalized derivative and got delta functions.

Naturally enough we call these two pieces the singular part and regular part of the the full generalized derivative.
Example 19. In Example 14 we found the generalized derivative was

$$
f^{\prime}(t)=2 \delta(t+2)-2 \delta(t)+3 \delta(t-3)+ \begin{cases}0 & \text { for } t<-2 \\ 1 & \text { for }-2<t<-1 \\ -1 & \text { for }-1<t<0 \\ x & \text { for } 0<t<1 \text { and } 1<t<2 \\ -6(x-2) & \text { for } 2<t\end{cases}
$$

To avoid any confusion between primes and subscripts let's denote $f^{\prime}(t)$ by $g(t)$. This is a generalized function. Its singular part is $g_{s}(t)=2 \delta(t+2)-2 \delta(t)+3 \delta(t-3)$ and its regular part is

$$
g_{r}(t)= \begin{cases}0 & \text { for } t<-2 \\ 1 & \text { for }-2<t<-1 \\ -1 & \text { for }-1<t<0 \\ x & \text { for } 0<t<1 \text { and } 1<t<2 \\ -6(x-2) & \text { for } 2<t\end{cases}
$$

The two parts make up the entire generalized function, i.e.

$$
f^{\prime}(t)=g(t)=g_{s}(t)+g_{r}(t) .
$$

## 4 Laplace transform and LTI systems with impulsive input

In simple systems with input $\delta(t)$ we could easily determine the post-initial conditions. But what about more complicated systems? For example,

$$
P(D) x=D \delta(t) \quad \text { with rest initial conditions. }
$$

or

$$
P(D) x=Q(D) \delta(t) \quad \text { with rest initial conditions. }
$$

The right hand side of these equations involves derivatives of $\delta(t)$. These are generalized derivatives, but we do not have a good physical or graphical intuition about how they affect the system. Fortunately, the Laplace transform allows us to easily solve these equations without this intuition.

### 4.1 The Laplace transform of the delta function

Theorem. $\mathcal{L}(\delta)=1$.
Proof. Since $\delta(t)=u^{\prime}(t)$, the theorem follows from our previous proof that $\mathcal{L}\left(u^{\prime}\right)=1$. For completeness, and because it's so easy, let's also prove this directly from the definition of Laplace transform:

$$
\mathcal{L}(\delta(t))=\int_{0^{-}}^{\infty} \delta(t) e^{-s t} d t=e^{0 \cdot t}=1
$$

Note. This proof shows why we needed to use $0^{-}$for the lower limit of the Laplace integral: the integral would not make sense if we used plain 0 for the limit.
Theorem. $\mathcal{L}(\delta(t-a))=e^{-a s}$ for any $a>0$.
Proof. Again we use the integral definition of Laplace transform:

$$
\mathcal{L}(\delta(t-a))=\int_{0^{-}}^{\infty} \delta(t-a) e^{-s t} d t=e^{-s a}
$$

Note that just as in the $t$-translation formula (also called the $t$-shift formula) in the table, the shift by $a$ on the time side resulted in a factor of $e^{-a s}$ on the frequency side.

### 4.2 The unit impulse response and the transfer function

Definition. For the LTI system with input $f(t)$ and response $x(t)$

$$
P(D) x=Q(D) f \quad \text { with rest initial conditions }
$$

the unit impulse response is the solution when the input $f(t)=\delta(t)$.
Theorem. The unit impulse response is the inverse Laplace transform of the transfer function.

Proof. The proof is a triviality: $\mathcal{L}(\delta)=1$. So, since output $=$ transfer $\times$ input, the response to input $\delta(t)$ has the transfer function as its Laplace transform.

### 4.3 Examples

Here we will solve several LTI equations with various inputs. First we will redo some of the earlier examples.

Example 20. Solve $\dot{x}+k x=\delta$ with rest initial conditions.
answer: The Laplace transform gives $X(s)=1 /(s+k)$ so $x(t)=e^{-k t}$ for $t>0$. The full solution is

$$
x(t)= \begin{cases}0 & \text { for } t<0 \\ e^{-k t} & \text { for } t>0\end{cases}
$$

We call this format of the solution cases format because we write a different formula for each of the cases $t<0$ and $t>0$.

Another convenient format is $u$-format. Here we make use of the fact that $u(t)$ is 0 for $t<0$ to write the above solution as

$$
x(t)=u(t) e^{-k t}
$$

Example 21. Find the unit impulse response for the operator $D+3 I$. Give your answer in both $u$ and cases format.
answer: The unit impulse response is the solution to

$$
(D+3 I) x=\dot{x}+3 x=\delta(t), \quad \text { with rest IC. }
$$

Taking the Laplace transform we get $X(s)=1 /(s+3)$. The inverse Laplace transform now gives $x(t)$ in cases-format and $u$-format as

$$
x(t)=\left\{\begin{array}{ll}
0 & \text { for } t<0 \\
e^{-3 t} & \text { for } t>0
\end{array}=u(t) e^{-3 t} .\right.
$$

Notes: 1. The post-initial condition is $x\left(0^{+}\right)=1$. This came out of the solution, we didn't have to think about the effect of the input $\delta(t)$ at $t=0$.
2. The Laplace transform method did not help us find $x(t)$ for $t<0$. For this we used the rest IC that are part of the definition of the unit impulse function.
3. Since $x\left(0^{-}\right)=0$ the output jumps by 1 unit at $t=0$.

Example 22. Find the unit impulse response for the system

$$
\left(D^{2}+2 D+2 I\right) x=D f
$$

where we consider $f$ to be the input and $x$ to be the response. Give your answer in both $u$ and cases format.
answer: The Laplace transform is $X(s)=s /\left(s^{2}+2 s+2\right)$. To use the Laplace table (or partial fractions) we have to complete the square:

$$
X(s)=\frac{s}{(s+1)^{2}+1}
$$

To apply the table we need to do a little algebraic manipulation:

$$
X(s)=\frac{s+1}{(s+1)^{2}+1}-\frac{1}{(s+1)^{2}+1}
$$

Now the table gives us $x(t)=e^{-t} \cos (t)-e^{-t} \sin (t)$. Therefore

$$
x(t)=\left\{\begin{array}{ll}
0 & \text { for } t<0 \\
e^{-t} \cos (t)-e^{-t} \sin (t) & \text { for } t>0
\end{array}=u(t)\left(e^{-t} \cos (t)-e^{-t} \sin (t)\right)\right.
$$

Note: The post-initial conditions emerge naturally from the solution and are $x\left(0^{+}\right)=$ $1, \dot{x}\left(0^{+}\right)=-2$.
Example 23. (Resonance) Assume rest initial conditions and solve the equation $\ddot{x}+x=f(t)$, where $f(t)=\delta(t)+\delta(t-2 \pi)+\delta(t-4 \pi)+\ldots$
answer: We have the Lapace transform

$$
\left(s^{2}+1\right) X(s)=1+e^{-2 \pi s}+e^{-4 \pi s}+e^{-6 \pi s}+\ldots
$$

So

$$
X(s)=\frac{1}{s^{2}+1}+\frac{e^{-2 \pi s}}{s^{2}+1}+\frac{e^{-4 \pi s}}{s^{2}+1}+\ldots
$$

We know $\mathcal{L}\left(1 /\left(s^{2}+1\right)=\sin (t)\right.$. So, the $t$-translation formula in the table shows

$$
\mathcal{L}^{-1}\left(\frac{e^{-2 n \pi s}}{s^{2}+1}\right)=u(t-2 n \pi) \sin (t-2 n \pi)=u(t-2 n \pi) \sin (t) .
$$

Therefore,

$$
\left.\begin{array}{rl}
x(t) & =(u(t)+u(t-2 \pi)+u(t-4 \pi)+u(t-6 \pi)+\ldots) \sin (t) \\
& = \begin{cases}0 & \text { for } t<0 \\
\sin (t) & \text { for } 0<t<2 \pi \\
2 \sin (t) & \text { for } 2 \pi<t<4 \pi \\
3 \sin (t) & \text { for } 4 \pi<t<6 \pi\end{cases} \\
& \ldots
\end{array}\right] .
$$

