

# Partial Fractions and the Coverup Method

## 18.031

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## 1 Heaviside Cover-up Method

### 1.1 Introduction

The cover-up method was introduced by Oliver Heaviside as a fast way to do a decomposition into partial fractions. This is an essential step in using the Laplace transform to solve differential equations, and this was more or less Heaviside's original motivation.

The cover-up method can be used to make a partial fractions decomposition of a proper rational function  $\frac{p(s)}{q(s)}$  whenever the denominator can be factored into distinct linear factors.

**Note:** We put this section first, because the coverup method is so useful and many people have not seen it. Some of the later examples rely on the full algebraic method of undetermined coefficients presented in the next section. If you have never seen partial fractions you should read that section first.

### 1.2 Linear Factors

We first show how the method works on a simple example, and then show why it works.

**Example 1.** Decompose  $\frac{s-7}{(s-1)(s+2)}$  into partial fractions.

**answer:** We know the answer will have the form

$$\frac{s-7}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}. \quad (1)$$

To determine  $A$  by the cover-up method, on the left-hand side we mentally remove (or cover up with a finger) the factor  $s-1$  associated with  $A$ , and substitute  $s=1$  into what's left; this gives  $A$ :

$$\left. \frac{s-7}{(s+2)} \right|_{s=1} = \frac{1-7}{1+2} = -2 = A. \quad (2)$$

Similarly,  $B$  is found by covering up the factor  $s+2$  on the left, and substituting  $s=-2$  into what's left. This gives

$$\left. \frac{s-7}{(s-1)} \right|_{s=-2} = \frac{-2-7}{-2-1} = 3 = B.$$

Thus, our answer is

$$\frac{s-7}{(s-1)(s+2)} = \frac{-2}{s-1} + \frac{3}{s+2}. \quad (3)$$

### 1.3 Why does the method work?

The reason is simple. The “right” way to determine  $A$  from Equation 1 would be to multiply both sides by  $(s - 1)$ ; this would give

$$\frac{s - 7}{(s + 2)} = A + \frac{B}{s + 2}(s - 1). \quad (4)$$

Now if we substitute  $s = 1$ , what we get is exactly Equation 2, since the term on the right with  $B$  disappears. The cover-up method therefore is just an easy and efficient way of doing the calculations.

In general, if the denominator of the proper rational function factors into the product of distinct linear factors:

$$\frac{p(s)}{(s - a_1)(s - a_2) \cdots (s - a_r)} = \frac{A_1}{s - a_1} + \cdots + \frac{A_r}{s - a_r}, \quad a_i \neq a_j,$$

then  $A_i$  is found by covering up the factor  $s - a_i$  on the left, and setting  $s = a_i$  in the rest of the expression.

**Example 2.** Decompose  $\frac{1}{s^3 - s}$  into partial fractions.

**answer:** Factoring,  $s^3 - s = s(s^2 - 1) = s(s - 1)(s + 1)$ . By the cover-up method,

$$\frac{1}{s(s - 1)(s + 1)} = \frac{-1}{s} + \frac{1/2}{s - 1} + \frac{1/2}{s + 1}.$$

To be honest, the real difficulty in all of the partial fractions methods (the cover-up method being no exception) is in factoring the denominator. Even the programs which do symbolic integration, like Macsyma, or Maple, can only factor polynomials whose factors have integer coefficients, or “easy coefficients” like  $\sqrt{2}$ . and therefore they can only integrate rational functions with “easily-factored” denominators.

### 1.4 Quadratic Factors

Heaviside’s cover-up method can be used even when the denominator doesn’t factor into distinct linear factors. This only gives partial results, but these can often be a big help, as the following example illustrates.

**If you have never seen partial fractions before you might want to read the next section on the algebraic method of undetermined coefficients.**

**Example 3.** Decompose  $\frac{5s + 6}{(s^2 + 4)(s - 2)}$ .

**answer:** We write

$$\frac{5s + 6}{(s^2 + 4)(s - 2)} = \frac{As + B}{s^2 + 4} + \frac{C}{s - 2}. \quad (5)$$

We first determine  $C$  by the cover-up method, getting  $C = 2$ . Then  $A$  and  $B$  can be found by the method of undetermined coefficients; the work is greatly reduced since we need to solve only two simultaneous equations to find  $A$  and  $B$ , not three.

Following this plan, using  $C = 2$ , we combine terms on the right of (5) so that both sides have the same denominator. The numerators must then also be equal, which gives us

$$5s + 6 = (As + B)(s - 2) + 2(s^2 + 4). \quad (6)$$

Comparing the coefficients of  $s^2$  and of the constant terms on both sides of (6) gives the two equations

$$0 = A + 2 \quad \text{and} \quad 6 = -2B + 8,$$

from which  $A = -2$  and  $B = 1$ .

In using Equation 6, one could have instead compared the coefficients of  $s$ , getting  $5 = -2A + B$ , leading to the same result, but providing a valuable check on the correctness of the computed values for  $A$  and  $B$ .

In Example 3, an alternative to undetermined coefficients would be to substitute two numerical values for  $s$  into the original Equation 5, say  $s = 0$  and  $s = 1$  (any values other than  $s = 2$  are usable). Again one gets two simultaneous equations for  $A$  and  $B$ . This method requires addition of fractions, and is usually better when only one coefficient remains to be determined (as in the example just below).

Still another method would be to factor the denominator completely into linear factors, using complex coefficients, and then use the cover-up method, but with complex numbers. At the end, conjugate complex terms have to be combined in pairs to produce real summands, and the calculations can sometimes be longer.

## 1.5 Repeated Linear Factors

The cover-up method can also be used if a linear factor is repeated, but there too it gives just partial results. It applies only to the *highest power of the linear factor*.

**Example 4.** Decompose  $\frac{1}{(s-1)^2(s+2)}$ .

**answer:** We write

$$\frac{1}{(s-1)^2(s+2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s+2}. \quad (7)$$

To find  $A$  cover up  $(s-1)^2$  and set  $s = 1$ ; you get  $A = 1/3$ . To find  $C$ , cover up  $s+2$ , and set  $s = -2$ ; you get  $C = 1/9$ .

This leaves  $B$  which cannot be found by the cover-up method. But since  $A$  and  $C$  are already known in Equation 7,  $B$  can be found by substituting any numerical value (other than 1 or  $-2$ ) for  $s$  in (7). For instance, if we put  $s = 0$  and remember that  $A = 1/3$  and  $C = 1/9$ , we get

$$\frac{1}{2} = \frac{1/3}{1} + \frac{B}{-1} + \frac{1/9}{2},$$

giving  $B = -1/9$ .

$B$  could also be found by applying the method of undetermined coefficients to the Equation 7; note that since  $A$  and  $C$  are known, it is enough to get a single linear equation in order to determine  $B$  — simultaneous equations are no longer needed.

The fact that the cover-up method works for just the *highest power* of the repeated linear factor can be seen just as before. In the above example for instance, the cover-up method for finding  $A$  is just a short way of multiplying Equation 5 through by  $(s - 1)^2$  and then substituting  $s = 1$  into the resulting equation.

## 2 Partial Fractions: Undetermined Coefficients

### 2.1 Introduction

Logically this section should precede the previous one on coverup since it explains what we are doing with partial fraction and shows an algebraic method that never fails. However, since most students in this course will have seen partial fractions before it seemed reasonable to start with the coverup method.

### 2.2 Rational Functions

A **rational function** is one that is the ratio of two polynomials. For example

$$\frac{s + 1}{s^2 + 7s + 9} \quad \text{and} \quad \frac{s^2 + 7s + 9}{s + 1}$$

are both rational functions.

A rational function is called **proper** if the degree of the numerator is strictly smaller than the degree of the denominator; in the examples above, the first is proper while the second is not.

**Long-division:** Using long-division we can always write an improper rational function as a polynomial plus a proper rational function. The partial fraction decomposition only applies to proper functions.

**Example 5.** Use long-division to write  $\frac{s^3 + 2s + 1}{s^2 + s - 2}$  as a the sum of a polynomial and a proper rational function.

**answer:**

$$s^2 + s - 2 \quad \left| \begin{array}{r} s - 1 \\ s^3 + 2s + 1 \\ \underline{s^3 + s^2 - 2s} \\ -s^2 + 4s + 1 \\ \underline{-s^2 - s + 2} \\ 5s - 1 \end{array} \right.$$

Therefore,

$$\frac{s^3 + 2s + 1}{s^2 + s - 2} = s - 1 + \frac{5s - 1}{s^2 + s - 2}.$$

### 2.3 Linear Factors

Here we assume the denominator factors in distinct linear factors. We start with a simple example. We will explain the general principle immediately afterwards.

**Example 6.** Decompose  $R(s) = \frac{s-3}{(s-2)(s-1)}$  using partial fractions. Use this to find  $\mathcal{L}^{-1}(R(s))$ .

**answer:**

$$\frac{s-3}{(s-2)(s-1)} = \frac{A}{s-2} + \frac{B}{s-1}.$$

Multiplying both sides by the denominator on the left gives

$$s-3 = A(s-1) + B(s-2) \tag{8}$$

The sure algebraic way is to expand out the right hand side and equate the coefficients with those of the polynomial on the left.

$$s-3 = (A+B)s + (-A-2B) \Rightarrow \begin{cases} \text{coeff. of } s: & 1 = A+B \\ \text{coeff. of } 1: & -3 = -A-2B \end{cases}$$

We solve this system of equations to find the undetermined coefficients  $A$  and  $B$ :  $A = -1$ ,  $B = 2$ .

Answer:  $R(s) = -1/(s-2) + 2/(s-1)$ .

Table lookup then gives  $\mathcal{L}^{-1}(R(s)) = -e^{2t} + 2e^t$ .

**Note:** In this example it would have been easier to plug the roots of each factor into Equation 8. When you do this every term except one becomes 0.

Plug in  $s = 1 \Rightarrow -2 = B(-1) \Rightarrow B = 2$

Plug in  $s = 2 \Rightarrow -1 = A(1) \Rightarrow A = -1$ .

In general, if  $P(s)/Q(s)$  is a proper rational function and  $Q(s)$  factors into distinct linear factors  $Q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$  then

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}.$$

The proof of this is not hard, but we will not give it. Remember you must have a *proper* rational function and each of the factors must be distinct. Repeated factors are discussed below.

**Example 7.** Use partial fractions to find  $\mathcal{L}^{-1}\left(\frac{3}{s^3-3s^2-s+3}\right)$ .

**answer:** The hardest part of this problem is to factor the denominator. For higher order polynomials this might be impossible. In this case you can check

$$s^3 - 3s^2 - s + 3 = (s-1)(s+1)(s-3).$$

The partial fractions decomposition is

$$\frac{3}{(s-1)(s+1)(s-3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s-3}.$$

Multiplying through by the denominator gives

$$3 = A(s+1)(s-3) + B(s-1)(s-3) + C(s-1)(s+1).$$

Plugging in  $s = 1$  gives  $A = -3/4$ , likewise  $s = -1$  gives  $B = 3/2$  and  $s = 3$  gives  $C = -3/4$ .

Our answer is

$$\mathcal{L}^{-1}\left(\frac{3}{s^3 - 3s^2 - s + 3}\right) = Ae^t + Be^{-t} + Ce^{3t} = -\frac{2}{4}e^t + \frac{3}{2}e^{-t} - \frac{3}{4}e^{3t}.$$

## 2.4 Quadratic Factors

If the denominator has quadratic factors then, the numerator in the partial fraction decomposition will be a linear term instead of a constant.

**Example 8.** Find  $\mathcal{L}^{-1}\left(\frac{s-1}{(s+1)(s^2+4)}\right)$ .

**answer:** This is a proper rational function so

$$\frac{s-1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}. \quad (9)$$

Notice the quadratic factor gets a linear term in the numerator. Notice also that the number of unknown coefficients is the same as the degree of the denominator in the original fraction.

From Equation 9 we can write

$$\mathcal{L}^{-1}\left(\frac{s-1}{(s+1)(s^2+4)}\right) = Ae^{-t} + B\cos(2t) + \frac{C}{2}\sin(2t).$$

All that's left is to do some algebra to find the coefficients. Multiplying Equation 9 through by the denominator gives

$$s-1 = A(s^2+4) + (Bs+C)(s+1) = (A+B)s^2 + (B+C)s + (4A+C).$$

Equate the coefficients on both sides:

$$\begin{aligned} s^2: & \quad 0 = A + B \\ s: & \quad 1 = B + C \\ s^0: & \quad -1 = 4A + C \end{aligned}$$

Solving, we get  $A = -2/5$ ,  $B = 2/5$ ,  $C = 3/5$ .

**Example 9.** Don't be fooled by quadratic terms that factor into linear ones.

$$\frac{1}{(s+1)(s^2-4)} = \frac{1}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2}.$$

**Example 10.** Don't forget that the rational function must be proper. For example, decompose  $\frac{s^3+2s+1}{s^2+s-2}$  using partial fractions.

**answer:** First, we must use long-division to make this proper. From Example (5) we have

$$\frac{s^3+2s+1}{s^2+s-2} = s-1 + \frac{5s-1}{s^2+s-2} = s-1 + \frac{5s-1}{(s+2)(s-1)} = s-1 + \frac{A}{s+2} + \frac{B}{s-1}.$$

Solving for the undetermined coefficients gives  $A = 11/3$ ,  $B = 4/3$ .

## 2.5 Repeated Linear Factors

For repeated linear factors we need one partial fraction term for each power of the factor as illustrated by the following example.

**Example 11.** Find  $\mathcal{L}^{-1}\left(\frac{2s}{s^3(s+1)^2(s+2)}\right)$  using partial fractions.

**answer:**

$$\frac{2s}{s^3(s+1)^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1} + \frac{E}{(s+1)^2} + \frac{F}{s+2}$$

Here the denominator has a linear factor  $s$  repeated three times (term  $s^3$ ), and a linear factor  $(s+1)$  repeated twice (term  $(s+1)^2$ ); hence three partial fractions are associated with the first, while two are associated with the latter. The term  $(s+2)$  which is not repeated leads to one partial fraction as previously seen. You can check that the coefficients are

$$A = -5/2, B = 1, C = 0, D = 2, E = 2, F = 1/2.$$

Using the  $s$ -shift rule we have  $\mathcal{L}^{-1}(1/(s+1)^2) = te^{-t}$ . Thus,

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{2s}{s^3(s+1)^2(s+2)}\right) &= A + Bt + \frac{C}{2}t^2 + De^{-t} + Ete^{-t} + Fe^{-2t} \\ &= -\frac{5}{2} + t + 2e^{-t} + 2te^{-t} + \frac{1}{2}e^{-2t}. \end{aligned}$$

## 2.6 Repeated Quadratic Factors

Just like repeated linear factors, quadratic factors have one term for each power of the factor as illustrated in the following example.

**Example 12.** Find  $\mathcal{L}^{-1}\left(\frac{2s}{s(s^2+1)^2(s^2+4s+2)}\right)$  using partial fractions.

**answer:** The partial fractions decomposition is

$$\frac{2s}{s(s^2+1)^2(s^2+4s+6)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{Ds+E}{(s^2+1)^2} + \frac{Fs+G}{s^2+4s+6}$$

Note the repeated factor  $(s^2+1)^2$  lead to two partial fraction terms.

We won't compute the coefficients –you can do this by going through the algebra. Instead, we'll focus on finding the Laplace inverse. Our table contains the terms with repeated quadratic factors.

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{Ds}{(s^2+1)^2}\right) &= D\frac{t}{2}\sin(t) \\ \mathcal{L}^{-1}\left(\frac{E}{(s^2+1)^2}\right) &= E\frac{1}{2}(\sin(t) - t\cos(t)). \end{aligned}$$

For the terms with denominator  $s^2+s+2$  we need to complete the square. Notice that we make sure to also shift the  $s$ -term in the numerator.

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{Fs+G}{s^2+4s+6}\right) &= \mathcal{L}^{-1}\left(\frac{F(s+2)+G-2F}{(s+2)^2+2}\right) \\ &= Fe^{-2t}\cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}(G-2F)e^{-2t}\sin(\sqrt{2}t). \end{aligned}$$

Putting this together, the answer is

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{2s}{s(s^2+1)^2(s^2+4s+6)}\right) &= A + B \cos(t) + C \sin(t) \\ &\quad + D \frac{t}{2} \sin(t) + E \frac{1}{2}(\sin(t) - t \cos(t)) \\ &\quad + F e^{-2t} \cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}(G - 2F)e^{-2t} \sin(\sqrt{2}t). \end{aligned}$$

## 2.7 Complex Factors

We can allow complex roots. In this case all quadratic terms factor into linear terms.

**Example 13.** Decompose  $s/(s^2 + \omega^2)$  using complex partial fractions and use it to show  $\mathcal{L}^{-1}(s/(s^2 + \omega^2)) = \cos(\omega t)$ .

**answer:**

$$\frac{s}{s^2 + \omega^2} = \frac{s}{(s - i\omega)(s + i\omega)} = \frac{A}{s - i\omega} + \frac{B}{s + i\omega}.$$

Multiplying through by the denominator gives  $s = A(s + i\omega) + B(s - i\omega)$ .

Plug in  $s = i\omega \Rightarrow A = 1/2$ .

Plug in  $s = -i\omega \Rightarrow B = 1/2$ .

From the table:

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + \omega^2}\right) = \mathcal{L}^{-1}\left(\frac{A}{s - i\omega} + \frac{B}{s + i\omega}\right) = A e^{i\omega t} + B e^{-i\omega t} = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) = \cos(\omega t).$$