Laplace transfom: t-translation rule
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1 Introductory example

Consider the system $\dot{x} + 3x = f(t)$, where f is the input and x the response. We know its unit impulse response is

$$w(t) = \begin{cases} 0 & \text{for } t < 0 \\ e^{-3t} & \text{for } t > 0 \end{cases} \quad = \quad u(t)e^{-3t}.$$

This is the response from rest IC to the input $f(t) = \delta(t)$. What if we shifted the impulse to another time, say, $f(t) = \delta(t-5)$? Linear time invariance tells us the response will also be shifted. That is, the solution to

$$\dot{x} + 3x = \delta(t-2), \qquad \text{with rest IC} \tag{1}$$

is

$$x(t) = w(t-2) = \begin{cases} 0 & \text{for } t < 2 \\ e^{-3t} & \text{for } t > 2 \end{cases} \quad = \quad u(t-2)e^{-3(t-2)}.$$

In words, this is a system of exponential decay. The decay starts as soon as there is an input into the system. Graphs are shown in Figure 1 below.

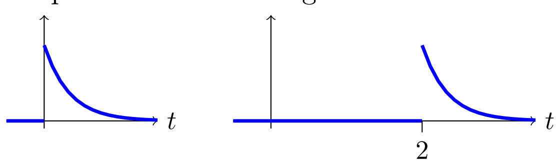

Figure 1. Graphs of $w(t)$ and $x(t) = w(t-2)$.

We know that $\mathcal{L}(\delta(t-a)) = e^{-as}$. So, we can find $X = \mathcal{L}(x)$ by taking the Laplace transform of Equation 1.

$$(s+3)X(s) = e^{-2s} \;\Rightarrow\; X(s) = \frac{e^{-2s}}{s+3} = e^{-2s}W(s),$$

where $W = \mathcal{L}w$. So delaying the impulse until $t = 2$ has the effect in the frequency domain of multiplying the response by e^{-2s}. This is an example of the t-translation rule.

2 t-translation rule

The t-translation rule, also called the t-shift rule gives the Laplace transform of a function shifted in time in terms of the given function. We give the rule in two forms.

$$\mathcal{L}(u(t-a)f(t-a);\, s) \quad = \quad e^{-as}F(s) \tag{2}$$
$$\mathcal{L}(u(t-a)f(t);\, s) \quad = \quad e^{-as}\mathcal{L}(f(t+a);\, s). \tag{3}$$

For completeness we include the translation formulas for $u(t-a)$ and $\delta(t-a)$:

$$\mathcal{L}(u(t-a)) \quad = \quad e^{-as}/s \tag{4}$$
$$\mathcal{L}(\delta(t-a)) \quad = \quad e^{-as}. \tag{5}$$

Remarks:
1. Formula 3 is ungainly. The notation will become clearer in the examples below.
2. Formula 2 is most often used for computing the inverse Laplace transform, i.e., as

$$u(t-a)f(t-a) = \mathcal{L}^{-1}\left(e^{-as}F(s)\right).$$

3. These formulas parallel the s-shift rule. In that rule, multiplying by an exponential on the time (t) side led to a shift on the frequency (s) side. Here, a shift on the time side leads to multiplication by an exponential on the frequency side.

Proof: The proof of Formula 2 is a very simple change of variables on the Laplace integral.

$$\begin{aligned}
\mathcal{L}(u(t-a)f(t-a);\, s) &= \int_0^\infty u(t-a)f(t-a)e^{-st}\,dt \\
&= \int_a^\infty f(t-a)e^{-st}\,dt \qquad (u(t-a)=0 \text{ for } t<a) \\
&= \int_0^\infty f(\tau)e^{-s(\tau+a)}\,d\tau \qquad (\text{change of variables:} \quad \tau = t-a) \\
&= e^{-as}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau \\
&= e^{-as}F(s).
\end{aligned}$$

Formula 3 follows easily from Formula 2. The easiest way to proceed is by introducing a new function. Let $g(t) = f(t+a)$, so

$$f(t) = g(t-a) \quad \text{and} \quad G(s) = \mathcal{L}(g) = \mathcal{L}(f(t+a)).$$

We get

$$\mathcal{L}(u(t-a)f(t);\, s) \;=\; \mathcal{L}(u(t-a)g(t-a)) \;=\; e^{-as}G(s) = e^{-as}\mathcal{L}(f(t+a);\, s).$$

The second equality follows by applying Formula 2 to $g(t)$.

Example 1. Find $\mathcal{L}^{-1}\left(\dfrac{\omega e^{-as}}{s^2+\omega^2}\right)$.

__**answer:**__ First ignore the exponential and let

$$f(t) = \mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin(\omega t).$$

Using the shift Formula 2 this becomes

$$\mathcal{L}^{-1}\left(\frac{\omega e^{-as}}{s^2+\omega^2}\right) = u(t-a)f(t-a) = u(t-a)\sin\omega(t-a).$$

Example 2. $\mathcal{L}\left(u(t-3)t;\, s\right) = e^{-3s}\mathcal{L}(t+3;\, s) = e^{-3s}\left(\dfrac{1}{s^2}+\dfrac{3}{s}\right)$.

Example 3. $\mathcal{L}(u(t-3)\cdot 1;\, s) = e^{-3s}\mathcal{L}(1;\, s) = e^{-3s}/s$.

Example 4. Find $\mathcal{L}(f)$ for $f(t) = \begin{cases} 0 & \text{for } t < 2 \\ t^2 & \text{for } t > 2. \end{cases}$

answer: In order to use the t-shift rule we have to write $f(t)$ in u-format:

$$f(t) = u(t-2)t^2.$$

So, Formula 3 says

$$\mathcal{L}(f) = e^{-2s}\mathcal{L}((t+2)^2;\, s) = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right).$$

Example 5. Find $\mathcal{L}(f)$ for $f(t) = \begin{cases} \cos(t) & \text{for } 0 < t < 2\pi \\ 0 & \text{for } t > 2\pi. \end{cases}$

answer: Again we first put $f(t)$ in u-format. Notice the the function

$$u(t) - u(t-2) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore

$$f(t) = (u(t) - u(t-2\pi))\cos(t) = u(t)\cos(t) - u(t-2\pi)\cos(t).$$

Using the t-translation formula 3 we get

$$\mathcal{L}(f) = \frac{s}{s^2+1} - e^{-2\pi s}\mathcal{L}(\cos(t+2\pi)) = \frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1}.$$

The last equality holds because $\cos(t+2\pi) = \cos(t)$.

3 A longer example

The fish population in a lake is not reproducing fast enough and the population is decaying exponentially with decay rate k. A program is started to stock the lake with fish. Three different scenarios are discussed below.

Example 6. A program is started to stock the lake with fish at a constant rate of r units of fish/year. Unfortunately, after 1/2 year the funding is cut and the program ends. Model this situation and solve the resulting DE for the fish population as a function of time.

answer: Let $x(t)$ be the fish population and let $A = x(0^-)$ be the initial population. Exponential decay means the population is modeled by

$$\dot{x} + kx = f(t), \quad x(0^-) = A \tag{6}$$

where $f(t)$ is the rate fish are being added to the lake. In this case

$$f(t) = \begin{cases} r & \text{for } 0 < t < 1/2 \\ 0 & \text{for } 1/2 < t. \end{cases}$$

First we write f in u-format: $f(t) = r\,(1 - u(t - 1/2))$ and find the Laplace transform of the equation.

$$F(s) = \mathcal{L}(f)(s) = \frac{r}{s} - \frac{r}{s}e^{-s/2}.$$

Next we find the Laplace transform of the equation and solve for f.

$$\begin{aligned} sX - x(0^-) + kX &= F(s) \\ (s+k)X - A &= \frac{r}{s}(1 - e^{-s/2}) \\ X(s) &= \frac{A}{s+k} + \frac{r}{s(s+k)}(1 - e^{-s/2}). \end{aligned}$$

To find $x(t)$ we temporarily ignore the factor of $e^{-s/2}$ and take Laplace inverse of what's left. (using partial fractions).

$$\mathcal{L}^{-1}\left(\frac{A}{s+k}\right) = Ae^{-kt}, \qquad \mathcal{L}^{-1}\left(\frac{r}{s(s+k)}\right) = \frac{r}{k}(1 - e^{-kt}).$$

The t-translation formula then says

$$\mathcal{L}^{-1}\left(\frac{re^{-s/2}}{s(s+k)}\right) = \frac{r}{k}u(t - 1/2)\left(1 - e^{-k(t-1/2)}\right).$$

Putting it all together we get (in u and cases format).

$$\begin{aligned} x(t) &= Ae^{-kt} + \frac{r}{k}\left(1 - e^{-kt}\right) - \frac{r}{k}u(t - 1/2)\left(1 - e^{-k(t-1/2)}\right) \\ &= \begin{cases} Ae^{-kt} + \frac{r}{k}\left(1 - e^{-kt}\right) & \text{for } 0 < t < 1/2 \\ Ae^{-kt} - \frac{r}{k}\left(e^{-kt} + e^{-k(t-1/2)}\right) & \text{for } 1/2 < t. \end{cases} \end{aligned}$$

Example 7. (Periodic on/off) The program is refunded and they have enough money to stock at a constant rate of r for the first half of each year. Find $x(t)$ in this case.

answer: All that's changed from Example 6is the input function $f(t)$. We write it in cases-format and translate that to u-format so we can take the Laplace transform.

$$\begin{aligned} f(t) &= \begin{cases} r & \text{for } 0 < t < 1/2 \\ 0 & \text{for } 1/2 < t < 1 \\ r & \text{for } 0 < t < 3/2 \\ 0 & \text{for } 3/2 < t < 2 \\ & \dots \end{cases} \\ &= r\left(1 - u(t - \frac{1}{2}) + u(t-1) - u(t - \frac{3}{2}) + \dots\right) \end{aligned}$$

The computations from here are essentially the same as in the previous example. We sketch them out.

$$\mathcal{L}(f) = \frac{r}{s}\left(1 - e^{-s/2} + e^{-s} - e^{-3s/2} + \dots\right), \quad \text{so} \quad X = \frac{A}{s+k} + \frac{r}{s(s+k)}\left(1 - e^{-s/2} + e^{-s} - \dots\right).$$

Thus,

$$\begin{aligned} x(t) =& Ae^{-kt} + \frac{r}{k}\Big[(1-e^{-kt}) - u(t-1/2)(1-e^{-k(t-1/2)}) \\ &+ u(t-1)(1-e^{-k(t-1)}) - u(t-3/2)(1-e^{-k(t-3/2)}) + \ldots\Big] \end{aligned}$$

and in cases format:

$$x(t) = \begin{cases} Ae^{-kt} + \frac{r}{k} - \frac{r}{k}e^{-kt} & \text{for } 0 < t < \frac{1}{2} \\ Ae^{-kt} - \frac{r}{k}\left(e^{-kt} - e^{-k(t-1/2)}\right) & \text{for } \frac{1}{2} < t < 1 \\ \ldots & \\ Ae^{-kt} + \frac{r}{k} - \frac{r}{k}\left(e^{-kt} - e^{-k(t-1/2)} + \ldots + e^{-k(t-n)}\right) & \text{for } n < t < n + \frac{1}{2} \\ Ae^{-kt} - \frac{r}{k}\left(e^{-kt} - e^{-k(t-1/2)} + \ldots - e^{-k(t-n-1/2)}\right) & \text{for } n + \frac{1}{2} < t < n+1 \\ \ldots & \end{cases}$$

Note that the pattern in the formula for the response alternates between the periods of stocking and not stocking. In particular, notice that the constant term r/k is only present during periods of stocking.

Example 8. (Impulse train) The answer to the previous example is a little hard to read. We know from experience that impulsive input usually leads to simpler output. In this scenario suppose that once a year $r/2$ units of fish are dumped all at once into the lake. Find $x(t)$ in this case.

answer: Once again, all that's changed from Example 6 is the input function $f(t)$. In this case we have

$$f(t) = \frac{r}{2}\left(\delta(t) + \delta(t-1) + \delta(t-2) + \delta(t-3) + \ldots\right).$$

This is called an impulse train. Its Laplace transform is easy to find.

$$F(s) = \frac{r}{2}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right).$$

One nice thing about delta functions is that they don't introduce any new terms into the partial fractions part of the problem.

$$\begin{aligned} sX(s) - x(0^-) + kX(s) &= \frac{r}{2}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right). \\ \Rightarrow\ X(s) &= \frac{A}{s+k} + \frac{r}{2(s+k)}\left(1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots\right). \end{aligned}$$

Laplace inverse is easy:

$$\mathcal{L}^{-1}\left(\frac{1}{s+k}\right) = e^{-kt} \qquad \Rightarrow \qquad \mathcal{L}^{-1}\left(\frac{e^{-ns}}{s+k}\right) = u(t-n)e^{-k(t-n)}.$$

Thus,

$$x(t) = Ae^{-kt} + \frac{r}{2}e^{-kt} + \frac{r}{2}u(t-1)e^{-k(t-1)} + \frac{r}{2}u(t-2)e^{-k(t-2)} + \frac{r}{2}u(t-3)e^{-k(t-3)} + \ldots$$

Here are graphs of the solutions to examples 6 and 8 (with $A = 0$, $k = 1$, $r = 2$). Notice how they settle down to periodic behavior.

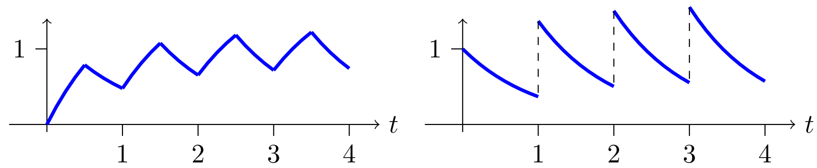

Fig. 1. Graphs from example 2 (left) and example 3 (right).