The Laplace Transform
18.031, Haynes Miller and Jeremy Orloff

1 Laplace transform basics: introduction

An operator takes a function as input and outputs another function. A transform does the same thing with the added twist that the output function has a *different* independent variable. The Laplace transform takes a function $f(t)$ and produces a function $F(s)$.

We will allow the variable s to be complex. As with the transfer function you learned about earlier the s will be thought of as complex frequency

You should think of $f(t)$ and $F(s)$ as two views of the same underlying object. If we have a signal, then $f(t)$ is the familiar view of that signal in time and $F(s)$ is the less familiar view in frequency. Everything about the signal is present in both views, but some things are easier to see in one view or the other. Using them together gives us a powerful tool for understanding systems and signals.

You have already been using the Laplace transform without knowing it. For the system

$$P(D)x = Q(D)f, \quad \text{(considering } f \text{ to be the input and } x \text{ the output)}$$

the transfer function $Q(s)/P(s)$ is a Laplace transform of a function $w(t)$. Over time we will give $w(t)$ several different names. To highlight its importance here we will call it the fundamental solution to the system. Also, just like the transfer function, the Laplace transform of any function has a pole diagram.

In practice the Laplace transform has the following benefits:

- It makes explicit the long-term behavior of $f(t)$.
- The pole diagram of $F(s)$ gives a succinct summary of some of the important properties of $f(t)$.
- It allows us to easily compute the transfer function for LTI systems other than $P(D)x = Q(D)f$. In particular, it will do this for systems with delay.
- Since it is the bridge between the time domain and the frequency domain we can credit it with all the benefits of using transfer functions:
  - It allows us to analyze LTI systems, and in particular block diagrams, using simple algebra.
  - For the transfer function, when we restrict s by setting it equal to $i\omega$ we get the frequency response of the system
  - The pole diagram of a transfer function can show at a glance the stability and frequency response of a system. It is an important engineering design tool.

2 Definition of Laplace transform

The Laplace transform of a function $f(t)$ of a real variable t is another function depending on a new variable s, which is in general complex. We will denote the Laplace transform of f by $\mathcal{L}f$. It is defined by the integral

$$(\mathcal{L}f)(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt, \tag{1}$$

for all values of s for which the integral converges.

There are a few things to note.

- $\mathcal{L}f$ is only defined for those values of s for which the improper integral on the right-hand side of (1) converges.
- We will allow s to be complex.
- The use of 0^-, in the definition (1) is necessary to accomodate what we'll call delta functions. Until we learn about such functions it will not be important. In those cases where 0^- isn't needed we will allow ourselves to use the less precise form

$$(\mathcal{L}f)(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt. \tag{1'}$$

- The limits of integration mean that the Laplace transform is only concerned with functions on $(0^-, \infty)$. What happens before time $t = 0^-$ does not play a role.

3 Notation, $F(s)$

We will adopt the following conventions:

1. Writing $(\mathcal{L}f)(s)$ can be cumbersome so we will often use an uppercase letter to indicate the Laplace transform of the corresponding lowercase function:

   $$(\mathcal{L}f)(s) = F(s), \qquad (\mathcal{L}g)(s) = G(s), \text{ etc.}$$

   For example, in the formula (see the Laplace table)

   $$\mathcal{L}(f') = sF(s) - f(0^-)$$

   it is understood that $F(s)$ means $\mathcal{L}(f)$.

2. Another notation we will use is $\mathcal{L}(f(t); s)$
3. If our function doesn't have a name we will use the formula instead. For example, the Laplace transform of the function t^2 can written $\mathcal{L}(t^2; s)$ or more simply $\mathcal{L}(t^2)$.
4. If in some context we need to modify $f(t)$, e.g. by applying a translation by a number a, we can write $\mathcal{L}(f(t-a); s)$ or $\mathcal{L}(f(t-a))$ for the Laplace transform of $f(t-a)$.
5. You've already seen several different ways to use parentheses. Sometimes we will even drop them altogether. So, if $f(t) = t^2$ then the following all mean the same thing

   $$(\mathcal{L}f; s) = F(s) = \mathcal{L}f(s) = \mathcal{L}(f(t); s) = \mathcal{L}(t^2; s); \quad \mathcal{L}f = F = \mathcal{L}(t^2).$$

4 Why s is called frequency

The Laplace transform variable s is thought of as complex frequency. We alreadt saw this in the transfer function: if $H(s)$ is the transfer function of an LTI system, then when $s = i\omega$ we have $H(s) = H(i\omega)$ is the complex gain of the system.

A couple of other small points are in order here. The first is that for the exponential e^{st} to make sense the exponent st must be dimensionless. Therefore the units of s must be 1/time, which are the same as the units of frequency.

The second point is really a reiteration of what happens when we used complex replacement to find the Sinusoidal Response Formula. Euler's formula says $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ and we call ω the angular frequency. By analogy we call any exponent s in e^{st} a complex frequency.

5 First examples

For the first few examples we will explicitly use a limit for the improper integral. Soon we will do this implicitly without comment.

Example 1. Let $f(t) = 1$, find $F(s) = \mathcal{L}f(s)$.

answer: Using the definition $(1')$ we have

$$\mathcal{L}(1) = F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \left.\frac{e^{-st}}{-s}\right]_0^T = \lim_{T\to\infty} \left.\frac{e^{-sT}-1}{-s}\right]_0^T.$$

The limit depends on whether the real part of s is positive or negative.

$$\lim_{T\to\infty} e^{-sT} = \begin{cases} 0 & \text{if } \mathrm{Re}(s) > 0 \\ \infty & \text{if } \mathrm{Re}(s) < 0. \end{cases}$$

Therefore,

$$\mathcal{L}(1) = F(s) = \begin{cases} \frac{1}{s} & \text{if } \mathrm{Re}(s) > 0 \\ \text{diverges} & \text{if } \mathrm{Re}(s) \le 0. \end{cases}$$

(We didn't actually compute the case $\mathrm{Re}(s) = 0$, but it is easy to see it diverges.)

Example 2. Compute $\mathcal{L}(e^{at})$.

answer: Using the definition $(1')$ we have

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}\,dt = \lim_{T\to\infty} \left.\frac{e^{(a-s)t}}{a-s}\right]_0^T = \lim_{T\to\infty} \left.\frac{e^{(a-s)T}-1}{a-s}\right]_0^T.$$

The limit depends on whether the real part of $a - s$ is positive or negative.

$$\lim_{T\to\infty} e^{(a-s)T} = \begin{cases} 0 & \text{if } \mathrm{Re}(s) > \mathrm{Re}(a) \\ \infty & \text{if } \mathrm{Re}(s) < \mathrm{Re}(a). \end{cases}$$

Therefore,

$$\mathcal{L}(e^{at}) = \begin{cases} \frac{1}{s-a} & \text{if } \mathrm{Re}(s) > \mathrm{Re}(a) \\ \text{diverges} & \text{if } \mathrm{Re}(s) \le \mathrm{Re}(a). \end{cases}$$

(We didn't actually compute the case $\mathrm{Re}(s) = \mathrm{Re}(a)$, but it is easy to see it diverges.)

We now have the first two entries in our table of Laplace transforms:

$$\begin{array}{llll} f(t) = 1 & \Rightarrow & F(s) = 1/s, & \mathrm{Re}(s) > 0 \\ f(t) = e^{at} & \Rightarrow & F(s) = 1/(s-a), & \mathrm{Re}(s) > \mathrm{Re}(a). \end{array}$$

Note that the last field in each line gives the range of s where the Laplace integral converges.

6 Linearity

You will not be surprised to learn that the Laplace transform is linear. For functions f, g and constants c_1, c_2

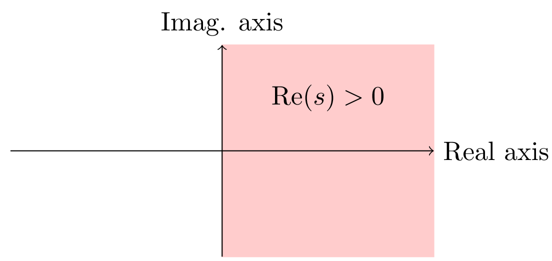
$$\mathcal{L}(c_1 f + c_2 g) = c_1 \mathcal{L}(f) + c_2 \mathcal{L}(g)$$

This is clear from the definition (1) of $\mathcal{L}$ because integration is linear.

7 Domain of $F(s)$: complex s and the region of convergence

As we've seen, we allow s to be complex and use, as needed, properties of the complex exponential.

Example 3. Previously we saw that $\mathcal{L}(1) = 1/s$, valid for all s with $\mathrm{Re}(s) > 0$. The region $\mathrm{Re}(s) > 0$ is called the region of convergence of the transform. It is a right half-plane in the complex s-plane.



Region of convergence: right half-plane $\mathrm{Re}(s) > 0$.

Likewise, the region of convergence for $f(t) = e^{at}$ is the right half-plane $\mathrm{Re}(s) > \mathrm{Re}(a)$. We will see that the region of convergence for any function is a right half-plane.

7.1 The domain of $F(s)$

For $f(t)$ we have $F(s) = 1/s$ with region of convergence $\mathrm{Re}(s) > 0$. But, the function $1/s$ is well defined for all $s \neq 0$. So we can extend the domain of $F(s)$ beyond the region of convergence to all $s \neq 0$.

The process of extending the domain of $F(s)$ beyond the region of convergence is called *analytic continuation*. Though it has wider applicability: In this class analytic continuation will always consist of extending $F(s)$ to the complex plane minus the zeros of the denominator.

8 Functions of exponential order and piecewise continuous functions

This is a technical section which assures us that the Laplace transform makes sense for all the functions we care about in 18.031

As we computed Laplace integrals we were careful to note for which values of s they converged. It can happen the integral does not converge for any value of s. In this case we say that the function fails to have a Laplace transform.

Example 4. It is easy to see that $f(t) = e^{t^2}$ has no Laplace transform.

The problem is the e^{t^2} grows too fast as t gets large.

8.1 Functions of Exponential Order

The class of functions that do have Laplace transforms are those of *exponential order*. We will see that every function of exponential order has a Laplace transform valid in a right half-plane $\text{Re}(s) > a$ for some value a.

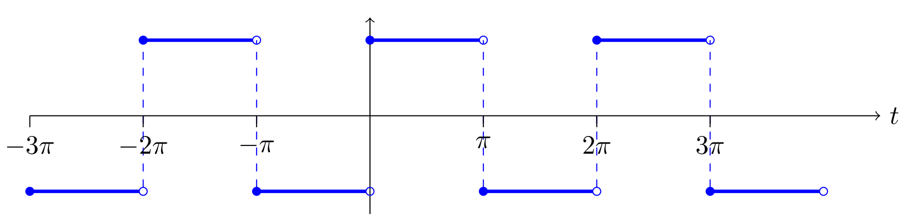
A function is said to be of exponential order if there are numbers a and M such that $|f(t)| < Me^{at}$. In this case, we say that f has exponential order a. We will give details below, but the basic idea is that in the Laplace integral the exponential decay of the e^{-st} term will compensate for the growth of $f(t)$ as long as $f(t)$ grows slower than some exponential. Fortunately for us, all the functions we use in this class are of this type.

Example 5. The functions 1, $\cos(\omega t)$, $\sin(\omega t)$, t^n all have exponential order 0. The function e^{at} has exponential order a.

8.2 Piecewise continuous functions

A function $f(t)$ is piecewise continuous if it is continuous everywhere except at a finite number of points in any finite interval and if at these points it has a jump discontinuity (i.e. a jump of finite height).

Example 6. The square wave is piecewise continuous.



The square wave is piecewise continuous.

The main point of this section is the following theorem which assures us that the Laplace transform converges for all the functions we use in 18.031.

Theorem: If $f(t)$ is piecewise continuous and of exponential order a then the Laplace transform $\mathcal{L}f(s)$ converges for all s with $\text{Re}(s) > a$.

Proof: Suppose $\text{Re}(s) > a$, so $s = (a + \alpha) + ib$ for some positive α. We are given that

$|f(t)| < Me^{at}$. So,

$$|f(t)e^{-st}| = |f(t)e^{-(a+\alpha)t}e^{-ibt}| = |f(t)e^{-(a+\alpha)t)}| < Me^{at}\mathrm{e}^{-(a+\alpha)t} = Me^{-\alpha t},$$

Here, we have used that $|\mathrm{e}^{-ibt}| = 1$.

Since $\int_0^\infty Me^{-\alpha t}\,dt$ converges for $\alpha > 0$, the Laplace transform integral also converges.

9 Partial fractions and inverse Laplace transform

In order to use the Laplace transform we need to be able to invert it and find $f(t)$ when we're given $F(s)$. Often this can be done by using the Laplace transform table. So for example, if $F(s) = 1/(s-5)$ then the table tells us that $f(t) = e^{5t}$.

More often we have to do some algebra to get $F(s)$ into a form suitable for the direct use of the table. Our main technique for doing this is the partial fractions decomposition. You probably used partial fractions in calculus as a method for computing integrals. If you need to relearn this technique we have posted a note on it. The note includes a description of the Heaviside coverup method. This is a simple and extremely useful computing device. If you do not know it already you should read the first section of the partial fraction note which explains it.

9.1 Laplace inverse by table lookup

We've added a few functions to our table of Laplace transforms. Soon we will add some more. But right now we will learn to use the table to find the inverse Laplace transform. We will illustrate this entirely by examples. Before we start you should open the copy of the table from the class website or else use the copy at the end of these notes.

Notation: The inverse Laplace transform will be denoted $\mathcal{L}^{-1}$.

Example 7. Find $\mathcal{L}^{-1}(1/(s-2))$.

answer: Use the table entry $\mathcal{L}(e^{at}) = 1/(s-a)$:

$$\mathcal{L}^{-1}(1/(s-2)) = e^{2t}.$$

Example 8. Find $\mathcal{L}^{-1}(1/(s^2+9))$.

answer: Use the table entry $\mathcal{L}(\sin(\omega t)) = \omega/(s^2+\omega^2)$ and linearity:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+3^2}\right) = \frac{1}{3}\sin(3t).$$

Example 9. Find $\mathcal{L}^{-1}(4/s^2)$.

answer: Use the table entry $\mathcal{L}(t) = 1/s^2$: $\mathcal{L}^{-1}(4/s^2) = 4t$.

Example 10. Find $\mathcal{L}^{-1}(4/(s-2)^2)$.

answer: We will use the s-shift formula $\mathcal{L}(e^{at}f(t)) = F(s-a)$. In this case we take $F(s) = 4/s^2$, which by Example 9 has $f(t) = t$. Therefore,

$$\mathcal{L}^{-1}(4/(s-2)^2) = \mathcal{L}^{-1}(F(s-2)) = e^{2t}f(t) = e^{2t}\,4t.$$

Example 11. Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+4s+13}\right)$.

answer: We first need to *complete the square*

$$s^2+4s+13 = s^2+4s+4+9 = (s+2)^2+9.$$

We have a shifted function $F(s+2)$, where $F(s) = 1/(s^2+9)$. Using Example 8, we know that $f(t) = \sin(3t)/3$, so using the s-shift rule we get

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4s+13}\right) = \mathcal{L}^{-1}(F(s+2)) = e^{-2t}\frac{\sin(3t)}{3}.$$

(Note that $\mathcal{L}(\omega/((s-a)^2+\omega^2)) = \mathrm{e}^{at}\sin(\omega t)$ is in the Laplace table, so we could have done this example directly.)

Example 12. Find $\mathcal{L}^{-1}\left(\dfrac{s}{(s^2+\omega^2)^2}\right)$.

answer: We haven't seen this formula yet, but there is a table entry, which gives: $\dfrac{t}{2\omega}\sin(\omega t)$.

Example 13. Find $\mathcal{L}^{-1}\left(\frac{1}{(s^2+\omega^2)^2}\right)$.

answer: This is also a table entry, answer: $\dfrac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))$.

10 More entries for the Laplace table

In this section we will add some new entries to our table of Laplace transforms.

Note: posted on the class website is the complete Laplace table that we will need in this class. For convenience it is also appended at the end of these notes.

10.1 Laplace transform of sine and cosine

1. $\mathcal{L}(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2}$, with region of convergence $\mathrm{Re}(s) > 0$.

2. $\mathcal{L}(\sin(\omega t)) = \dfrac{\omega}{s^2+\omega^2}$, with region of convergence $\mathrm{Re}(s) > 0$.

Proof: We know that

$$\cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}, \text{ and } \sin(\omega t) = \frac{e^{i\omega t}-e^{-i\omega t}}{2i}$$

(These formulas are extremely useful, if they are not immediately familiar you should take a moment to understand them. You can prove them by expanding each of the exponentials into $\cos() + i\sin()$.)

We already know that $\mathcal{L}(e^{at}) = 1/(s-a)$. Using this and the formulas above, we obtain

$$\mathcal{L}(\cos(\omega t)) = \mathcal{L}\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right) = \frac{1}{2}\left(\frac{1}{s-i\omega}+\frac{1}{s+i\omega}\right) = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}(\sin(\omega t)) = \mathcal{L}\left(\frac{e^{i\omega t}-e^{-i\omega t}}{2i}\right) = \frac{1}{2i}\left(\frac{1}{s-i\omega}-\frac{1}{s+i\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

The region of convergence follows from the fact that $\cos(\omega t)$ and $\sin(\omega t)$ both have exponential order 0.

Another approach would have been to use integration by parts to compute the transforms directly from the Laplace integral.

10.2 Laplace transform of derivatives

10.3 t-derivative rule

This course makes heavy use of differential equations, so we should try to compute $\mathcal{L}(f')$. (We use the notation f' instead of $\dot{f}$ simply because we think the dot does not sit nicely over the tall letter f.)

As usual, we write $\mathcal{L}(f;s) = F(s)$. The t-derivative rule is

$$\mathcal{L}(f') = sF(s) - f(0^-) \quad (2)$$

$$\mathcal{L}(f'') = s^2F(s) - sf(0^-) - f'(0^-) \quad (3)$$

$$\mathcal{L}(f^{(n)}) = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) + \ldots + f^{(n-1)}(0^-). \quad (4)$$

Technically we need to assume that $f(t)$ has exponential order a and $\mathrm{Re}(s) > a$. With this assumption $e^{-st}f(t)$ is 0 when $t = \infty$.

Proof: Rule (2) is a simple consequence of the definition of Laplace transform and integration by parts. The integration by parts formula is

$$\int uv'\,dt = uv - \int u'v\,dt.$$

We'll use this with $u = e^{-st}$ and $v' = f'(t)$, so that $u' = -se^{-st}$ and $v = f(t)$. Now, by definition $\mathcal{L}(f') = \int_{0^-}^{\infty} f'(t)e^{-st}\,dt$. We start by writing the definition of $\mathcal{L}(f')$ and then we apply integration by parts:

$$\mathcal{L}(f') = \int_{0^-}^{\infty} f'(t)e^{-st}\,dt = f(t)e^{-st}\Big]_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt = -f(0^-) + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

The last equality follows from our assumption that $f(t)e^{-st}$ is 0 at $t = \infty$. Now notice that the integral in the expression at the right is none other than $sF(s)$. Thus we have proved the t-derivative law $\mathcal{L}(f') = sF(s) - f(0^-)$.

Rule (3) follows by applying rule (2) twice.

$$\begin{aligned}\mathcal{L}(f'') &= s\mathcal{L}(f') - f'(0^-) \\ &= s(s\mathcal{L}(f) - f(0^-)) - f'(0^-) \\ &= s^2F(s) - sf(0^-) - f'(0^-).\end{aligned}$$

Rule (4) Follows by applying rule (2) n times.

Notes: 1. Calculations will be easiest when $f(0^-) = 0$, $f'(0^-) = 0$, etc. We will call this rest initial conditions.
2. A good way to think of the t-derivative rules is

$$\begin{aligned}\mathcal{L}(f) &= F(s) \\ \mathcal{L}(f') &= sF(s) + \text{ terms at } 0^-. \\ \mathcal{L}(f'') &= s^2F(s) + \text{terms at } 0^-.\end{aligned}$$

Roughly speaking, Laplace transforms differentiation in t to multiplication by s.

Example 14. Let $f(t) = e^{at}$. We can compute $\mathcal{L}(f')$ in two ways: directly and by using rule (2). Let's compute both ways and check that they give the same answer.

Directly: $f'(t) = ae^{at} \Rightarrow \mathcal{L}(f') = a/(s-a)$.

Rule (2): $\mathcal{L}(f) = F(s) = 1/(s-a) \Rightarrow \mathcal{L}(f') = sF(s) - f(0^-) = s/(s-a) - 1 = a/(s-a)$.
Both methods give the same answer.

Example 15. Let $f(t) = t^2+2t+1$. Compute $\mathcal{L}(f'')$ both directly and using the t-derivative rule.

answer: Directly: $f''(t) = 2 \Rightarrow \mathcal{L}(f'') = 2/s$.

Using rule (4): $\mathcal{L}(f'') = s^2F(s) - sf(0^-) - f'(0^-) = s^2(2/s^3 + 2/s^2 + 1/s) - s\cdot 1 - 2 = 2/s$.

Both methods give the same answer.

11 s-derivative rule

There is a certain symmetry in our formulas. If derivatives in time lead to multiplication by s then multiplication by t should lead to derivatives in s. This is true, but, as usual, there are small differences in the details of the formulas.

The s-derivative rule is

$$\mathcal{L}(tf;s) = -F'(s) \tag{5}$$
$$\mathcal{L}(t^nf;s) = (-1)^nF^{(n)}(s). \tag{6}$$

Proof: Rule (5) is a simple consequence of the definition of Laplace transform.

$$\begin{aligned} F(s) &= \mathcal{L}(f) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \\ \Rightarrow F'(s) &= \frac{d}{ds}\int_{0^-}^{\infty} f(t)e^{-st}\,dt \\ &= \int_{0^-}^{\infty} -tf(t)e^{-st} \\ &= -\mathcal{L}(tf(t)). \end{aligned}$$

Rule (6) is just rule (5) applied n times.

11.1 Powers of t and repeated factors in s

Now that we have the derivative rules we can use them to avoid having to compute a number of integrals. We derive more formulas for our Laplace table in a series of examples.

Example 16. Use the s-derivative rule to find $\mathcal{L}(t)$, $\mathcal{L}(t^2)$, etc.

answer: Start with $f(t) = 1$, then $F(s) = 1/s$. The s-derivative rule now says $\mathcal{L}(t) = -F'(s) = 1/s^2$

Using $\mathcal{L}(t) = 1/s^2$ we get

$$\mathcal{L}(t^2) = \mathcal{L}(t \cdot t) = -\frac{d\mathcal{L}(t)}{ds} = -(1/s^2)' = 2/s^3.$$

Continuing we get $\mathcal{L}(t^3) = -(2/s^3)' = 3 \cdot 2/s^4$. In general, we have $\mathcal{L}(t^n) = n!/s^{n+1}$.

Notes: 1. You should check that this rule gives the correct answer when $n = 1$

2. The region of convergence for $\mathcal{L}(t^n)$ is the same as for $\mathcal{L}(1)$, i.e. when $\mathrm{Re}(s) > 0$. The easiest way to see this is to notice that multiplying by powers of t does not change the exponential order of a function.

Example 17. Use the s-derivative rule to find $\mathcal{L}(te^{at})$ and $\mathcal{L}(t^n e^{at})$.

answer: Start with $f(t) = e^{at}$, then $F(s) = 1/(s-a)$. The s-derivative rule now says $\mathcal{L}(te^{at}) = -F'(s) = 1/(s-a)^2$.

Continuing:

$$\begin{aligned} &\mathcal{L}(t^2e^{at}) = F''(s) = 2/(s-a)^3 \\ &\mathcal{L}(t^3e^{at}) = -F'''(s) = 3\cdot 2/(s-a)^4 \\ &\mathcal{L}(t^4e^{at}) = F^{(4)}(s) = 4\cdot 3\cdot 2/(s-a)^5 \\ &\qquad\qquad \ldots \\ &\mathcal{L}(t^ne^{at}) = (-1)^nF^{(n)}(s) = n!/(s-a)^{n+1}. \end{aligned}$$

11.2 Repeated quadratic factors

Look at the table entries for repeated quadratic factors

$$\mathcal{L}\left(\frac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))\right) = \frac{1}{(s^2+\omega^2)^2} \tag{7}$$

$$\mathcal{L}\left(\frac{t}{2\omega}\sin(\omega t)\right) = \frac{s}{(s^2+\omega^2)^2} \tag{8}$$

$$\mathcal{L}\left(\frac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))\right) = \frac{s^2}{(s^2+\omega^2)^2} \tag{9}$$

Let's prove them using the s-derivative rule.

Proof of (8). Let $f(t) = \sin(\omega t)$. We know that $F(s) = \dfrac{\omega}{s^2+\omega^2}$, so the s-derivative rule implies

$$\mathcal{L}(t\sin\omega t) = -F'(s) = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

This is the same as formula (8) except the factor of 2ω is moved from one side to the other.

The other two formulas can be proved in a similar fashion. We won't give the proofs here.

11.3 s-shift formula

If a is any complex number and $f(t)$ is any function then the s-shift formula is

$$\mathcal{L}(e^{at}f(t)) = F(s-a).$$

Proof. As usual we write $F(s) = \mathcal{L}(f;s)$. If the region of convergence for $\mathcal{L}(f)$ is $\text{Re}(s) > a$ then the region of convergence for $\mathcal{L}(e^{at}f(t))$ is $\text{Re}(s) > Re(a) + a$.

Now we simply calculate directly from the definition of Laplace transform:

$$\begin{aligned}\mathcal{L}(e^{at}f(t)) &= \textstyle\int_0^\infty e^{at}f(t)e^{-st}\,dt \\ &= \textstyle\int_0^\infty f(t)e^{-(s-a)t}\,dt \\ &= F(s-a).\end{aligned}$$

Example 18. Find the Laplace transform of $e^{-t}\cos(3t)$.

<u>**answer:**</u> Since $\mathcal{L}(\cos(3t)) = s/(s^2+9)$ the s-shift formula gives

$$\mathcal{L}(\mathrm{e}^{-t}\cos(3t) = \frac{s+1}{(s+1)^2+9}.$$

Note: we could do this almost as easily by using Euler's formula to write

$$e^{-t}\cos(3t) = (1/2)\left(e^{(-1+3i)t} + e^{(-1-3i)t}\right).$$

We record here two important cases of the s-shift formula:

$$\begin{aligned}\mathcal{L}(e^{at}\cos(\omega t)) &= \frac{s-z}{(s-a)^2+\omega^2} \\ \mathcal{L}(e^{at}\sin(\omega t)) &= \frac{\omega}{(s-a)^2+\omega^2}.\end{aligned}$$

Consistency.
It is a good exercise to check for consistency among our various formulas:

1. We have $\mathcal{L}(1) = 1/s$, so the s-shift formula gives $\mathcal{L}(e^{at} \cdot 1) = 1/(s-a)$. This matches our formula for $\mathcal{L}(e^{at})$.
2. We have $\mathcal{L}(t^n) = n!/s^{n+1}$. If $n = 0$ we have $\mathcal{L}(t^0) = 0!/s = 1/s$. This matches our formula for $\mathcal{L}(1)$.

12 Laplace: solving initial value problems

12.1 Introduction

We now have everything we need to solve initial value problems using the Laplace transform. We will show how to do this through a series of examples.

To be honest we should admit that some initial value problems are more easily solved by other techniques. However, there are cases where the Laplace machinery can help keep things straight.

12.2 Examples of solving initial value problems (IVPs)

Example 19. Solve $\dot{x} + 3x = e^{-t}$ with rest initial conditions (rest IC).

answer: Rest IC mean that $x(t) = 0$ for $t < 0$, so $x(0^-)$, $\dot{x}(0^-)$, ... are all 0. As usual, we let $X = \mathcal{L}(x)$.

Using the t-derivative rule we can take the Laplace transform of (both sides) of the DE.

$$(sX(s) - x(0^-)) + 3X(s) = 1/(s+1).$$

Next we substitute the known value $x(0^-) = 0$ and solve for $X(s)$

$$(s+3)X(s) = \frac{1}{s+1} \;\Rightarrow\; X(s) = \frac{1}{(s+1)(s+3)}. \tag{10}$$

Finally, we find $x(t) = \mathcal{L}^{-1}(X)$ by using cover-up to do the partial fractions decomposition.

$$X(s) = \frac{1}{(s+1)(s+3)} = \frac{1/2}{s+1} - \frac{1/2}{s+3}, \quad \text{thus} \quad x(t) = \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \quad \text{for } t > 0.$$

Notes: 1. The term $e^{-t}/2$ is what the exponential response formula would give us. The term $e^{-3t}/2$ is the homogenous part of the solution, needed to match the IC.

2. This technique found $x(t)$ for $t > 0$. The rest IC tell us $x(t) = 0$ for $t < 0$.

3. The factor of $1/(s+3)$ in the expression for $X(s)$ in (10) is none other than the transfer function of the system: $\dot{x} + 3x = f(t)$.

Example 20. Solve $\dot{x} + 3x = e^{-t}$, $x(0^-) = 4$.

answer: Laplace:

$$sX(x) - x(0^-) + 3X(s) = 1/(s+1) \;\Rightarrow\; (s+3)X(s) = 4 + 1/(s+1).$$

Solve for $X(s)$:

$$X(s) = \frac{4}{s+3} + \frac{1}{(s+1)(s+3)} \tag{11}$$

We can use the partial fractions work from Example 19.

$$\begin{aligned} x(t) &= 4e^{-3t} + \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \quad \text{for } t > 0 \\ &= \frac{1}{2}e^{-t} + \frac{7}{2}e^{-3t} \quad \text{for } t > 0. \end{aligned}$$

Notes: (Same remarks as in the previous example.)

Example 21. Solve $\dot{x} + 2x = 4t$, with initial condition $x(0^-) = 1$.

answer: Taking the Laplace transform of the equation: $sX - x(0^-) + 2X = 4/s^2$.
Therefore,

$$X(s) = \frac{4}{s^2(s+2)} + \frac{1}{s+2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{1}{s+2}.$$

Therefore the inverse Laplace transform gives us

$$x(t) = A + Bt + Ce^{-2t} + e^{-2t}, \text{ for } t > 0.$$

The coverup method gets us $B = 2$, $C = 1$. Some algebra (undetermined coefficients) gives $A = -1$. Finally we record the exact solution:

$$x(t) = -1 + 2t + 2e^{-2t}, \text{ for } t > 0.$$

Example 22. Solve $\ddot{x} + 4x = \cos(2t)$, with rest initial conditions

answer: Laplace: $(s^2 + 4)X(s) = s/(s^2 + 4) \Rightarrow X(s) = s/(s^2 + 4)^2$. This is a repeated quadratic factor and it is in our table: $x(t) = t\sin(2t)/4$.

Notes:

1. This is a response of pure resonance.

2. We could have turned the logic around and used our previous knowledge of the solution to this equation to give yet another proof for the table entry $\mathcal{L}(t\sin(\omega t)/2\omega) = s/(s^2+\omega^2)^2$.

13 The transfer function

Consider the system with input $f(t)$ and response x:

$$P(D)x = Q(D)f.$$

In class we defined the transfer function as $Q(s)/P(s)$. This arose for us as when we tried solving the above system with input $f(t) = e^{st}$. In that case, the Exponential Response Formula gave us a particular solution $x(t) = (Q(s)/P(s))e^{st}$. Now that we have the Laplace transform we will see how it gives us the transfer function naturally and for arbitrary f.

Problem. Use the Laplace transform to solve the equation $P(D)x = Q(D)f$ starting with rest initial conditions.

answer: Let $X = \mathcal{L}(x)$ and $F = \mathcal{L}(f)$. Since we start from rest there are no 0^- terms, so we get $P(s)X(s) = Q(s)F(s)$, or equivalently

$$X(s) = \frac{P(s)}{Q(s)} F(s). \tag{12}$$

There we have it! Viewed in the frequency domain, the output is simply the transfer function times the input. We could say that the transfer function 'transfers' the input to the output. Another standard way of saying this is

$$\text{transfer function} = \frac{\text{output}}{\text{input}}.$$

Be sure to recognize that this is all taking place on the frequency side. Also be sure to marvel at the fact that from this side the output/input always gives the same function.

Here's a summary of what we have:

1. For the system $P(D)x = Q(D)f$ the transfer function is $W(s) = Q(s)/P(s)$.

2. In the frequency domain, solving for X with rest initial conditions is purely algebraic: $X(s) = W(S)F(s)$.

14 Block diagrams

14.1 Introduction

We already discussed some simple block diagrams when we introduced the notions of system, input, and output. Here, we will look at systems with more complicated block diagrams and show how to use them to compute the transfer functions.

As we do this, it will be useful to keep in mind the desciption of the transfer function as multiplying the input to get the output.

14.2 Simple examples

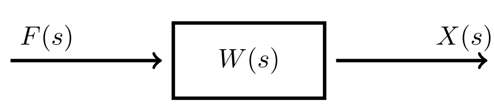
Example 23. Suppose we have the system $m\ddot{x}+b\dot{x}+kx = f(t)$, with input $f(t)$ and output $x(t)$. The Laplace transform converts this to functions and equations in the frequency variable s:

$$X(s) = \frac{1}{ms^2 + bs + k} F(s).$$

The transfer function for this system is $W(s) = 1/(ms^2 + bs + k)$, and we can write the relation between input and output as

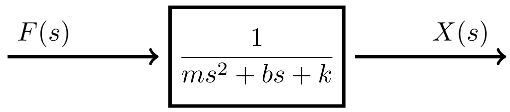
$$\text{input } F(s) \rightsquigarrow \text{ output } X(s) = W(s)F(s)$$

As a block diagram we can represent the system by



Block diagram for a system with transfer function $W(s)$.

Sometimes we write the formula for the transfer function in the box representing the system. For the above example this would look like



Block diagram giving the formula for the transfer function.

Example 24. (Cascading systems) Consider the cascaded system

$$P_1(D)x = Q_1(D)f, \quad P_2(D)y = Q_2(D)x, \quad \text{rest IC.}$$

The input to the cascade is f and the output is y. That is, the first equation takes the input f and outputs x. Then x is the input to the second equation, which ouputs y.

This is easy to solve on the frequency side. Let $W_1(s) = Q_1(s)/P_1(s)$ and $W_2(s) = Q_2(s)/P_2(s)$ be the transfer functions for the two differential equations. Considering the two equations separately we have

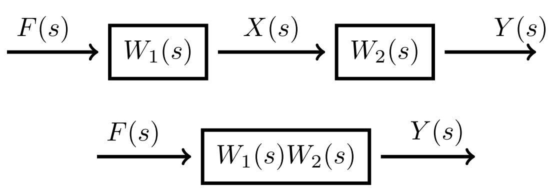
$$X(s) = W_1(s) \cdot F(s) \quad \text{and} \quad Y(s) = W_2(s) \cdot X(s).$$

It follows immediately that $Y(s) = W_2(s) \cdot W_1(s) \cdot F(s)$. Therefore the transfer function for the cascade is

$$\text{output/input} = Y(s)/F(s) = W_2(s) \cdot W_1(s).$$

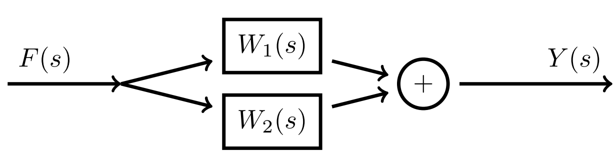
In other words, for cascaded systems the transfer functions multiply.

Representing this as block diagrams we have two equivalent diagrams



Equivalent block diagrams for a cascaded system.

Example 25. (Parallel systems) Suppose that we have a system consisting of two systems in parallel as shown in the block diagram.

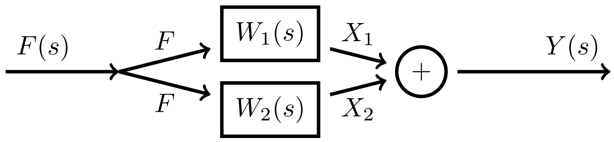


Systems in parallel.

Find the transfer function for the entire system.

answer: The plus sign in the circle indicates the two signals coming into the junction should be added. The split near the start indicates the same input $F(s)$ is sent into each system.

The way to figure out the transfer function is to name the outputs of each individual system.



System with intermediate outputs labeled.

For each system we know output = transfer function × input. Thus, $X_1 = W_1 \cdot F$, $X_2 = W_2 \cdot F$, $Y = X_1 + X_2$. So, we easily compute

$$Y = X_1 + X_2 = W_1 \cdot F + W_2 \cdot F = (W_1 + W_2) \cdot F.$$

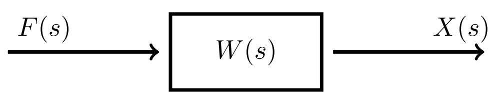
Therefore the transfer function is $W_1 + W_2$.

Example 26. An example of a parallel system is several microphones feeding the same sound into a mixing desk which in turn feeds an amplifier and speaker system.

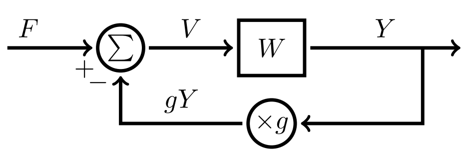
14.3 Feedback loops

Many systems are best described with feedback loops. In a broad sense, in a feedback loop the output is used to modify the input which in turn produces the output. In feedback control, the output of the system is monitored and used to modify the input in such a way as to ultimately produce a desired output. It is very hard to control a system without a feedback loop. For example, imagine trying to walk without sensory feedback about your surroundings. For most people, just shutting their eyes makes it hard to stand on one foot for very long.

Block diagrams are an excellent way to describe feedback. Suppose we start with a system with transfer function $W(s)$.



and modify it to have the feedback loop shown in the next figure.



The original system is known as the open loop system and the corresponding system with feedback is known as the closed loop system.

We've labeled the outputs from each system element. The symbol for the system element indicates what it does to its input(s). The symbol $\times g$ means the input to that element is scaled by g, that is apply a gain of g to the input. The symbol $\sum$ means the two inputs are combined; the plus and minus signs indicate whether to add or subtract the corresponding input.

The method of finding the transfer function is the same as in the previous examples. A bit of algebra gives

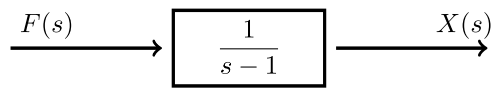
$$V = F - gY, \quad Y = W \cdot V \quad \Rightarrow \quad Y = W(F - gY) \Rightarrow \quad Y = \frac{W}{1 + gW} \cdot F.$$

As usual, the transfer function is output/input $= Y/F = W/(1+gW)$. This formula is one case of what is often called Black's formula.

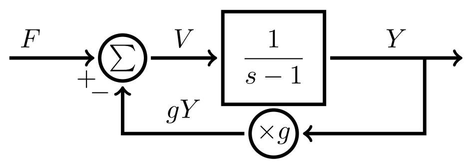
Example 27. Suppose we have an open loop system, say a circuit, with transfer function $W(s) = s/(as^2+bs+c)$. If we add a feedback loop with gain g then using Black's formula the closed loop transfer function is

$$\frac{s/(as^2+bs+c)}{1+gs/(as^2+bs+c)} = \frac{s}{as^2+(b+g)s+c}.$$

Example 28. Feedback can turn an unstable system into a stable one. Consider the open loop system with transfer function $1/(s-1)$.



This has a pole at $s = 1$, so it is unstable. Find all the values of a constant gain g that make the closed loop system stable.



<u>**answer:**</u> Black's formula tells us that the closed loop transfer function is

$$\frac{1/(s-1)}{1+g/(s-1)} = \frac{1}{s-1+g}.$$

Thus the closed loop system has one pole and it's at $1-g$. As long as $g > 1$ this pole is negative and the closed loop system is stable.

Note. The opposite is also true: feedback can make a stable system unstable.

15 Stability of a function

In this discussion of stability we will assume $\mathcal{L}(f)$ is a rational function. It is possible to deal with time shifted functions, but it would unduly complicate the presentation, so in this section we will assume that all functions have a Laplace transform of the form $F(s) = Q(s)/P(s)$.

15.1 Definition of stability for a function

We say that a function is exponentially stable if it goes to zero faster than some decaying exponential. Formally: $f(t)$ is exponentially stable if it is of negative exponential order, i.e. if

$$|f(t)| < Me^{-at}$$

for some positive M and a. Exponential stability is equivalent to all the poles of f have negative real part.

For a rational function $F(s) = Q(s)/P(s)$ a pole is called simple if it is not a repeated root of $P(s)$. If all the poles have negative real part except for at least one simple pole with zero real part, then we say that $f(t)$ is marginally stable. In this case, $f(t)$ may not decay to 0, but it stays bounded.

Example 29. The functions e^{-3t}, $e^{-3t}\cos(5t)$, $t^3 e^{-2t}$ are all exponentially stable.

Example 30. The functions $\cos(t)$, 1, $e^{-2t}+1$ are all marginally stable

Example 31. The function $t\sin(\omega t)$ has a double pole at $s = i\omega$. It is not stable or marginally stable and it is not bounded as t gets large.

Example 32. The function e^{2t} has a pole at $s = 2$. It is not bounded and not stable.

15.2 The final value theorem

We are often interested in the long term behavior of a system response. If we are working in the frequency domain it helps to have a way of determining the long term behavior directly from the Laplace transform of a function. The next theorem is useful in this regard.

Theorem. Final value theorem. Let $f(t)$ be a function and $F(s) = \mathcal{L}(f)$. If all the poles of $sF(s)$ have negative real part then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

An equivalent way of stating the criterion is that all the poles of $F(s)$ have negative real part except for possibly a simple pole at $s = 0$.

Proof. Rather than give a formal proof we will indicate an easy way to understand this theorem. The criterion that the poles of $sF(s)$ have negative real part means that the partial fraction decomposition of $F(s)$ is of the form

$$F(s) = \frac{A}{s} + \frac{B_1}{s+a_1} + \frac{B_2}{s+a_2} + \ldots + \frac{C_1}{(s+a_1)^2} + \ldots + \frac{D_1}{(s+a_1)^3} + \ \ldots,$$

where $\mathrm{Re}(a_j) < 0$. This means that $f(t)$ has the form

$$f(t) = A + B_1 e^{-a_1 t} + B_2 e^{-a_2 t} + \ldots + C_1 t e^{-a_1 t} + \ldots + D_1 t^2 e^{-a_1 t} + \ldots$$

Only the first of these terms does not go to 0 as t grows large. That is,

$$\lim_{t\to\infty} f(t) = A. \tag{13}$$

On the frequency side we have

$$sF(s) = A + \frac{B_1 s}{s+a_1} + \frac{B_2 s}{s+a_2} + \ldots \ \frac{C_1 s}{(s+a_1)^2} + \ldots, \frac{D_1 s}{(s+a)^3} + \ \ldots,$$

Only the first of these terms does not go to 0 as s goes to 0. That is,

$$\lim_{s\to 0} sF(s) = A. \tag{14}$$

Taken together Equations 13 and 14 prove the final value theorem.

Example 33. The function $f(t) = 2 + e^{-3t} + te^{-5t}$ has Laplace transform $F(s) = 2/s + 1/(s+3) + 1/(s+5)^2$. It has simple poles at $s = 0$ and $s = -3$ and a double pole at $s = -5$. If satisfies the criterion of the final value theorem and we can check:

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 2 + e^{-3t} + te^{-5t} = 2$$

and

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} 2 + \frac{s}{s+3} + \frac{s}{(s+5)^2} = 2.$$

Thus we have verified the theorem in this case.

Example 34. The function $f(t) = 2 + e^{-3t} + e^{5t}$ has Laplace transform $F(s) = 2/s + 1/(s+3) + 1/(s-5)$. It has simple poles at $s = 0$, $s = -3$ and $s = 5$. Because one of its poles is positive it *does not* satisfies the criterion of the final value theorem. Checking the two limits:

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 2 + e^{-3t} + e^{5t} = \infty$$

and

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} 2 + \frac{s}{s+3} + \frac{s}{s-5} = 2.$$

We see the limits are not the same! This shows us a valuable lesson:

Lesson. You must check that the hypothesis of the final theorem hold, otherwise you can get a misleading result.

Here's another example showing this lesson.

Example 35. The function $f(t) = \sin(2t)$ has Laplace transform $F(s) = 2/(s^2+4)$. It has simple poles at $s = \pm 2i$. These poles do not satisy the criterion of the final value theorem and we see

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \sin(2t) \text{ does not exist}$$

and

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s^2+4} = 0.$$

Again, the limits are not the same, but this does not violate the final value theorem because it does not apply to this case.

A good example of the use of the final value theorem is in the note on PID controllers posted on the class website.

Laplace Transform Table
18.031, January 2015

Properties and Rules

Function	**Transform**	
$f(t)$	$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$	(Definition)
$a\,f(t) + b\,g(t)$	$a\,F(s) + b\,G(s)$	(Linearity)
$e^{at}f(t)$	$F(s-a)$	(s-shift)
$f'(t)$	$sF(s) - f(0^-)$	
$f''(t)$	$s^2F(s) - sf(0^-) - f'(0^-)$	
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0^-) - \cdots - f^{(n-1)}(0^-)$	
$tf(t)$	$-F'(s)$	
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	
$u(t-a)f(t-a)$	$e^{-as}F(s)$	(t-translation or t-shift)
$u(t-a)f(t)$	$e^{-as}\mathcal{L}(f(t+a))$	(t-translation)
$(f * g)(t) = \int_{0^-}^{t^+} f(t-\tau)\,g(\tau)\,d\tau$	$F(s)\,G(s)$	
$\int_{0^-}^{t^+} f(\tau)\,d\tau$	$\frac{F(s)}{s}$	(integration rule)

Interesting, but not included in this course.

$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(\sigma)\,d\sigma$	

(The function table is on the next page.)

(The function table is on the next page.)

Function Table

Function	**Transform**	**Region of convergence**
1	$1/s$	$\mathrm{Re}(s) > 0$
e^{at}	$1/(s-a)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
t	$1/s^2$	$\mathrm{Re}(s) > 0$
t^n	$n!/s^{n+1}$	$\mathrm{Re}(s) > 0$
$\cos(\omega t)$	$s/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$\sin(\omega t)$	$\omega/(s^2+\omega^2)$	$\mathrm{Re}(s) > 0$
$e^{at}\cos(\omega t)$	$(s-a)/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$e^{at}\sin(\omega t)$	$\omega/((s-a)^2+\omega^2)$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\delta(t)$	1	all s
$\delta(t-a)$	e^{-as}	all s
$\cosh(kt) = \dfrac{e^{kt}+e^{-kt}}{2}$	$s/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\sinh(kt) = \dfrac{e^{kt}-e^{-kt}}{2}$	$k/(s^2-k^2)$	$\mathrm{Re}(s) > k$
$\dfrac{1}{2\omega^3}(\sin(\omega t) - \omega t\cos(\omega t))$	$\dfrac{1}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{t}{2\omega}\sin(\omega t)$	$\dfrac{s}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$\dfrac{1}{2\omega}(\sin(\omega t) + \omega t\cos(\omega t))$	$\dfrac{s^2}{(s^2+\omega^2)^2}$	$\mathrm{Re}(s) > 0$
$u(t-a)$	e^{-as}/s	$\mathrm{Re}(s) > 0$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$

Interesting, but not included in this course.

$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$	$\mathrm{Re}(s) > 0$
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$\mathrm{Re}(s) > 0$