Convolution
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1 Introduction

The convolution product of two functions is a peculiar looking integral which produces another function. It is found in a wide range of applications, so it has a special name and a special symbol. The convolution of f and g is denoted $f * g$ and defined by

$$(f * g)(t) = \int_{0^-}^{t^+} f(s)g(t-s)\,ds.$$

We will start studying this formula without any motivation. It's main properties are relatively easy to deduce from its definition.

The motivation will come in the form of Green's formula. Green's formula is an important tool which tells us how to solve a linear time invariant (LTI) system with any input and rest IC once we know its unit impulse response (weight function). We have already such a formula in the frequency domain, i.e. $X(s) = G(s)F(s)$. Green's formula is an equivalent formula, but completely in the time domain.

We know that many computations are more complicated in the time domain than in the frequency domain. So it is not surprising that Green's formula which involves convolution feels much more, well, convoluted than the simple formula $X(s) = G(s)F(s)$. We consider it because it offers some new insights into solutions of LTI systems and because it can be useful in cases where we don't know the Laplace transform of the input, but do know it in the time domain.

Technical Detail: Because we want convolution to work with delta functions we needed to be careful with the limits of integration. This explains the plus and minus on the limits. If both functions are continuous or have at most jump discontinuities then the limits can safely be set to 0 and t.

2 The missing formula

Let's make a quick summary of what we've learned about systems and the connections in between the time and frequency domains. Suppose we have a system $P(D)x = Q(D)f$. Then we know:

Frequency domain (s)	$\overset{\mathcal{L}}{\longleftarrow}$ $\underset{\mathcal{L}^{-1}}{\longrightarrow}$	Time domain (t)
System function $G(s) = \dfrac{Q(s)}{P(s)}$	$\longleftrightarrow$	weight function $w(t)$
Response $X(s) = G(s)F(s)$	$\longleftrightarrow$	Response $x(t) = ???$

We see that $X(s)$ is given by a formula in terms $G(s)$ and $F(s)$. So we expect that the missing formula in the last cell in the box should give $x(t)$ in terms of $w(t)$ and $f(t)$. In what follows below we will define the convolution of two functions; give it a notation $f * g$ and show the formula

$$x(t) = w * f(t).$$

3 Definition of convolution

We start by defining convolution. We will leave this unmotivated for a few sections, and for now just learn how to work with it.

The convolution of two functions f and g is a third function which we denote $f * g$. It is defined as the following integral

$$(f * g)(t) = \int_{0^-}^{t^+} f(\tau) g(t-\tau)\, d\tau \quad \text{for } t > 0. \tag{1}$$

There are a few things to point out about the formula.

- The variable of integration is τ. We can't use t because that is already used in the limits and in the integrand. We can choose any symbol we want for the variable of integration –it is just a *dummy* variable.
- The limits of integration are 0^- and t^+. This will only be important, when we work with delta functions. If f and g are continuous or have at worst jump discontinuities then we can use 0 and t for the limits.
- If we don't need to worry about delta functions we will often write convolution without the plus and minus:
$$f * g(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau.$$
- We are considering **one-sided convolution**. There is also a two-sided convolution where the limits of integration are $\pm\infty$.
- (**Important.**) One-sided convolution is only concerned with functions on the interval $(0^-, \infty)$. When using convolution we never look at $t < 0$.

4 Examples

Example 1 below calculates two useful convolutions from the definition (1). As you will see, the form of $f * g$ is not easy to guess from the forms of f and g.

Example 1. Show (i) $e^{at} * e^{bt} = \dfrac{e^{at} - e^{bt}}{a-b}$ if $a \neq b$; and (ii) $e^{at} * e^{at} = t\, e^{at}$.

<u>**answer:**</u> We will show (i); the calculation for (ii) is similar. If $a \neq b$,

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt} \int_0^t e^{(a-b)\tau}\, d\tau = e^{bt} \left. \frac{e^{(a-b)\tau}}{a-b} \right]_0^t = e^{bt} \frac{e^{(a-b)t} - 1}{a-b} = \frac{e^{at} - e^{bt}}{a-b}.$$

Note that because the functions are continuous we could safely integrate just from 0 to t instead of having to use limits 0^- and t^+.

One of our goals is to see that we can use convolution to give a formula for the response of an LTI system in terms of the weight function and input. The next example illustrates this for a first order differential equation.

Example 2. Use the variation of parameters formula to give an integral solution to the following first order constant-coefficient differential equation with rest initial conditions.

$$\dot{x} + kx = f(t). \tag{2}$$

Then rearrange the integral to show that gives the integral for the convolution of the input $f(t)$ and the weight function for this system.

For this example you can assume $f(t)$ is continuous so we don't have to worry about 0^- and t^+.

answer: Since $f(t)$ is left unspecified, the best we can hope for is a formula in terms of f. The variation of parameters formula gives

$$x(t) = \mathrm{e}^{-kt} \int_0^t f(\tau)\mathrm{e}^{k\tau}\, d\tau.$$

Notes: **1.** We won't derive this formula, but if you don't know it you can easily look it up.
2. Because the integral is easy to differentiate you can easily check the formula gives a solution by substituting it into the differential equation.
3. If you take $t = 0$ the integral 0 because it is over the interval $[0, 0]$, so $x(0) = 0$ satisfies the rest-initial condition.

Next we rewrite the solution to see it is a convolution. The system $\dot{x}+kx = f(t)$ has system function $1/(s + k)$. Thus the weight function is $w(t) = \mathcal{L}^{-1}(1/(s + k)) = u(t)\mathrm{e}^{-kt}$.

Now, since the integral is over τ the factor of e^{-kt} can be brought inside the integral. We get

$$x(t) = \mathrm{e}^{-kt} \int_0^t f(\tau)\mathrm{e}^{k\tau}\, d\tau. = \int_0^t f(\tau)\mathrm{e}^{-kt}\mathrm{e}^{k\tau}\, d\tau. = \int_0^t f(\tau)\mathrm{e}^{-k(t-\tau)}\, d\tau.$$

The last expression is a convolution. In fact, since $w(t) = \mathrm{e}^{-kt}$, we have that the last integral is just $w * f(t)$. That is, we have found the formula $x(t) = w * f(t)$.

This is the simplest case of **Green's formula**. It gives the response as the convolution of the input and weight functions. We will see that this holds for all LTI systems.

5 Properties of convolution

1. **Linearity:** Convolution is linear. That is, for functions f_1, f_2, g and constants c_1, c_2 we have

   $$(c_1f_1 + c_2f_2) * g = c_1(f_1 * g) + c_2(f_2 * g).$$

   This follows from the exact same property for integration. This might also be called the **distributive law**.

2. **Commutivity:** $f * g = g * f$.
   Proof: This follows directly by making the change of variable $v = t - \tau$. Limits: when $\tau = 0^-$ we have $v = t - \tau = t - 0^- = t^+$. Likewise, when $\tau = t^+$ we have $v = t - t^+ = 0^-$. Making the change of variables we get

$$(f * g)(t) = \int_{0^-}^{t^+} f(\tau)g(t-\tau)\,d\tau = \int_{0^-}^{t^+} f(t-v)g(v)\,dv = (g * f)(t)$$

3. **Associativity:** $f * (g * h) = (f * g) * h$. The proof just amounts to changing the order of integration in a double integral (left as an exercise).

6 Convolution with delta functions

Our goal in this section is to show that

$$(\delta * f)(t) = f(t) \quad \text{and} \quad (\delta(t-a) * f)(t) = f(t-a). \tag{3}$$

The notation for the second equation is ugly, but its meaning is clear.

We prove these formulas by direct computation. First, remember the rules of integration with delta functions: for $b > 0$

$$\int_{0^-}^{b} \delta(\tau)f(\tau)\,d\tau = f(0).$$

Now the formulas follow easily for $t > 0$:

$$\begin{aligned}(\delta * f)(t) &= \textstyle\int_{0^-}^{t^+} \delta(\tau) * f(t-\tau)\,d\tau = f(t-0) = f(t)\\ (\delta(t-a) * f)(t) &= \textstyle\int_{0^-}^{t^+} \delta(\tau-a) * f(t-\tau)\,d\tau = f(t-a).\end{aligned}$$

7 Convolution is a type of multiplication

You should think of convolution as a type of multiplication of functions. In fact, it is often referred to as the convolution product. In fact, it has the properties we associate with multiplication:

- It is commutative.
- It is associative.
- It is distributive over addition.
- It has a multiplicative identity. For ordinary multiplication, 1 is the multiplicative identity. Formula (3) shows that $\delta(t)$ is the multiplicative identity for the convolution product.

8 Green's Formula

In this section we state Green's formula for general LTI systems and look at some examples. We will prove it below.

Suppose that we have the linear time invariant system with rest IC.

$$P(D)x = Q(D)f(t), \quad x(t) = 0 \text{ for } t < 0 \tag{4}$$

- As always, we will consider $f(t)$ to be the input to this system.
- In this context, where we don't consider functions for $t < 0$, the initial conditions mean that $x(t)$ and all its derivatives are 0 at $t = 0^-$.

Theorem. Let $w(t)$ be the weight function for (4). Then, for any input $f(t)$ the solution to equation (4) is given by **Green's formula**

$$x(t) = (w * f)(t) = \int_{0^-}^{t^+} f(\tau)w(t-\tau)\, d\tau. \tag{5}$$

This is a wonderful formula! It tells us the response to *any* input once we know the unit impulse response. Furthermore, it gives us that response as an integral which can be computed numerically if necessary. For many physical systems the impulse response can be measured directly or deduced from measurements. So, Green's formula gives us a method for predicting the system's response to any input.

8.1 Examples using Green's formula

We now try out Green's formula (5) in a couple of cases where it can be checked against the solution found using another method.

Example 3. Let $f(t) = A$ be constant input to the system below with rest IC. Compute the response given by (5) and check this answer by solving the DE directly.

$$\ddot{x} + x = f(t), \quad x(0) = 0,\, \dot{x}(0) = 0.$$

<u>**answer:**</u> The system function is $1/(s^2+1)$ so the weight function is $w(t) = u(t)\sin(t)$, i.e. $w(t) = \mathcal{L}^{-1}(1/(s^2+1))$. Therefore for $t > 0$, we have

$$x(t) = w * f(t) = \int_0^t A\sin(t-\tau)\, d\tau = A\cos(t-\tau)\Big]_0^t = A(1-\cos(t)).$$

We check this by directly solving the differential equation $\ddot{x} + x = A$. It's easy to see that $x_p(t) = A$ is a particular solution. So the general solution is particular plus homogoneous, i.e.

$$x(t) = A + c_1\cos(t) + c_2\sin(t).$$

You can easily compute that the rest initial conditions are matched by the solution $x(t) = A - A\cos(t)$, exactly as found by Green's formula!

Example 4. Use Green's formula (5) to find the response of the following system for $t > 0$. As usual, assume rest IC.

$$x'' + x = f(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq \pi \\ 0 & \text{elsewhere} \end{cases}$$

<u>**answer:**</u> As in the previous example we have $w(t) = \sin(t)$. The convolution integral has two cases: $0 \leq t \leq \pi$ and $t \geq \pi$:

$$x(t) = \int_0^t f(\tau)\sin(t-\tau)\,d\tau = \begin{cases} \int_0^t \sin(t-\tau)\,d\tau = \cos(t-\tau)\Big]_0^t = 1 - \cos t, & \text{for } \ 0 \leq t \leq \pi; \\ \int_0^\pi \sin(t-\tau)\,d\tau = \cos(t-\tau)\Big]_0^\pi = -2\cos t, & \text{for } \ t \geq \pi\ . \end{cases}$$

We leave it to you to check this by solving the DE. You can do this directly or by using Laplace transform methods.

9 Building Green's formula from scratch

Now we will give a physical exponential decay example. Here we can build the solution from scratch using only methods from 18.01. This will give you a sense that Green's formula arises naturally.

Example 5. (The build up of a pollutant in a lake)
Every good formula deserves a particularly illuminating example, and perhaps the following will serve for the convolution integral. It is also illustrated by the Convolution: Accumulation Mathlet: `http://mathlets.org/mathlets/convolution-accumulation/`

Problem: We have a lake, and a pollutant is being dumped into it, at a certain variable rate $f(t)$ in kg/year. This pollutant degrades exponentially over time with decay rate k. If the lake begins at time zero with no pollutant, how much is in the lake at time $t > 0$?

<u>**answer:**</u> Let $x(t)$ be the amount of pollutant in the lake at time t. The model for this model is the familiar one of exponential decay

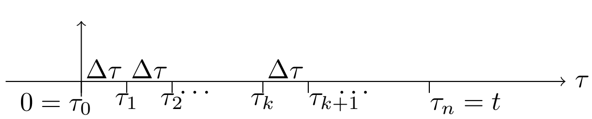
$$\dot{x} + kx = f(t),$$

We will solve for $x(t)$ using the 18.01 methods of slicing the time interval $[0, t]$ into tiny pieces; computing the contribution of each piece based on a *basic formula* and totaling the contributions of all the pieces using an integral.

First the basic formula of exponential decay: If an amount of pollutant A is thrown in the lake it will start to decay. After T years some of the orginal A will have decayed away. The amount left will be

$$x(t) = Ae^{-kT} \tag{6}$$

In our system pollutant is not being added all at once. Rather, it is dripping continuously into the lake. Following 18.01 we slice the interval $[0, t]$ into n small pieces of width $\Delta\tau$ as shown.



Let A_k be the amount of pollutant added in the interval $[\tau_k, \tau_{k+1}]$. Since $\Delta\tau$ is small and $f(\tau)$ is the rate pollutant is being added, we get the approximation

$$A_k \approx f(\tau_k)\Delta\tau.$$

But, we are interested in the contribution A_k makes to the total amount of pollutant in the lake at time t. Since it's tossed in the lake at time τ_k by time t it will have decayed for $t - \tau_k$ years. So by our basic equation of exponential decay. the amount of A_k left by time t will be

$$A_k \mathrm{e}^{-k(\tau_k - t)} = f(\tau_k)\Delta\tau \mathrm{e}^{-k(t-\tau_k)}.$$

This is approximately the contribution to $x(t)$ from the interval $[\tau_k, \tau_{k+1}]$. To determine the $x(t)$ we simply sum up the contributions of all the intervals.

$$\begin{aligned} x(t) &\approx A_1 e^{-a(t-\tau_1)} + \ldots + A_{n-1} e^{-a(t-\tau_{n-1})} \\ &\approx \left(f(\tau_1) e^{-a(t-\tau_1)} + \ldots + f(\tau_n) e^{-a(t-\tau_n)} \right) \Delta\tau. \end{aligned}$$

This is a Riemann sum. Taking the limit as $\Delta\tau \to 0$ we get the integral

$$x(t) = \int_0^t f(\tau) e^{-a(t-\tau)}\, d\tau. \tag{7}$$

Finally, notice that the weight function for this system is $w(t) = u(t)\mathrm{e}^{-kt}$ so the integral is indeed the convolution integral for $(w * f)(t)$.

10 Proof of Green's formula

Green's Formula: For the linear time invariant equation

$$P(D)x = Q(D)f(t), \qquad \text{with rest IC: } x(t) = 0 \text{ for } t < 0 \tag{8}$$

the solution for $t > 0$ is given by

$$x(t) = (w * f)(t) = \int_{0^-}^{t^+} f(\tau) w(t-\tau)\, d\tau, \tag{9}$$

where $w(t)$ is the weight function (unit impulse response) for the system.

Proof: The proof of Green's formula is surpisingly direct. We will use the linear time invariance of the system combined with superposition and the definition of the integral as a limit of Riemann sums.

To avoid worrying about 0^- and t^+ we will assume that $f(t)$ is continuous. With appropriate care, the proof will work for an $f(t)$ that has jump discontinuities or contains delta functions. We start by reminding ourselves of some basic facts about LTI systems.

Time invariance tells us that since $w(t)$ is the response to input $\delta(t)$ then $w(t-a)$ is the response to input $\delta(t-a)$.

Linearity or superposition tells us that the response to the linear combination $f(t) = c_1\delta(t) + c_2\delta(t-a)$ is $x(t) = c_1 w(t) + c_2\delta(t-a)$.

Now we will essentially repeat the argument in the polluted lake example to should that any input signal $f(t)$ can be thought of as a superposition of impulse.

First we partition the time axis into slices of width Δt. So, $t_0 = 0,\ t_1 = \Delta t,\ t_2 = 2\Delta t$, etc.

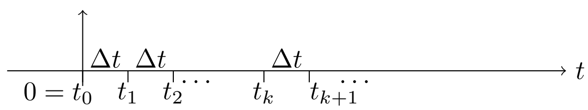


Figure 1: Division of the t-axis into small intervals.

Next we decompose the input signal $f(t)$ into packets over each interval. The kth signal packet, $f_k(t)$ coincides with $f(t)$ between t_k and t_{k+1} and is 0 elsewhere

$$f_k(t) = \begin{cases} f(t) & \text{for } t_k < t < t_{k+1} \\ 0 & \text{elsewhere.} \end{cases}$$

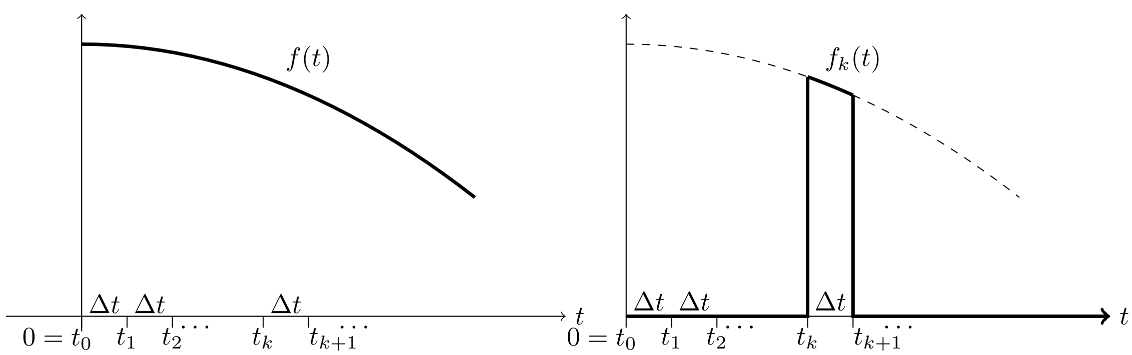


Figure 2: The signal packet $f_k(t)$.

It is clear that for $t > 0$ we have $f(t)$ is the sum of the packets

$$f(t) = f_0(t) + f_1(t) + \ldots + f_k(t) + \ldots$$

A single packet $f_k(t)$ is concentrated entirely in a small neighborhood of t_k so it is approximately an impulse with the same size as the area under $f_k(t)$. The area under $f_k(t) \approx f(t_k)\,\Delta t$. Hence,

$$f_k(t) \approx (f(t_k)\,\Delta t)\,\delta(t - t_k).$$

As we noted above: because of linear time invariance the response to $f_k(t)$ is

$$x_k(t) \approx (f(t_k)\,\Delta t)\,w(t - t_k).$$

We want to find the response at a fixed time. Since t is already in use, we will let T be our fixed time and find $x(T)$.

Since f is the sum of f_k, superposition gives x is the sum of x_k. That is, at time T

$$\begin{aligned} x(T) &= x_0(T) + x_1(T) + \ldots \\ &\approx \Big(f(t_0)w(T - t_0) + f(t_1)w(T - t_1) + \ldots\Big)\,\Delta t \end{aligned} \tag{10}$$

However, we can ignore all the terms when $t_k > T$. This is because the input packet $f_k(t)$ has its spike after time T, so it is at rest until after time T, which means so is the response $x_k(t)$. Now, if n is the last index where $t_k < T$ we have

$$x(T) \approx \Big(f(t_0)w(T-t_0) + f(t_1)w(T-t_1) + \ldots + f(t_n)w(T-t_n) \Big)\, \Delta t$$

This is a Riemann sum and as $\Delta t \to 0$ it goes to an integral

$$x(T) = \int_0^T x(t)w(T-t)\,dt$$

Except for the change in notation this is Green's formula (9).

11 Brief notes

Note on Causality: Causality is the principle that the future does not affect the past. Green's formula shows that the system (8) is causal. That is, the output $x(t)$ only depends on the input up to time t. Real physical systems are causal.

There are non-causal systems. For example, an audio compressor that gathers information after time t before deciding how to compress the signal at time t is non-causal. Another example is the system with input $f(t)$ and output $x(t)$ where x is the solution to $\dot{x} = f(t+1)$.

Example 6. (Resonance
Use Green's formula to solve the DE with rest inital conditions:

$$2\ddot{x} + 8x = \cos(2t), \quad x(0^-) = 0,\, \dot{x}(0^-) = 0$$

answer: The system function is $G(s) = 1/(2s^2+8)$, so the weight function is

$$w(t) = \mathcal{L}^{-1}(G(s)) = \frac{1}{4}u(t)\sin(2t).$$

Green's theorem then gives

$$x(t) = \frac{1}{4}\int_0^t \sin(2(t-\tau))\cos(2\tau)\,d\tau.$$

This is an easy integral, we sketch the algebra to compute it. It uses the trigonometric identity: $\sin(A)\cos(B) = \dfrac{\sin(A+B)+\sin(A-B)}{2}$.

$$\begin{aligned} x(t) &= \tfrac{1}{4}\textstyle\int_0^t \sin(2(t-\tau))\cos(2\tau)\,d\tau. \\ &= \tfrac{1}{8}\textstyle\int_0^t \left(\sin(2t)+\sin(2t-4\tau)\right)\,d\tau \\ &= \tfrac{1}{8}\left(\tau\sin(2t) + \tfrac{\cos(2t-4\tau)}{4}\right]_0^t \\ &= \frac{t\sin(2t)}{8}. \end{aligned}$$

This is the answer that you found in 18.03 using complex replacement and the extended exponential response formula.

12 Convolution in time and frequency

Now that we have Green's formula we can fill out the table from above.

Frequency domain (s)	$\xleftarrow{\mathcal{L}}$ $\xrightarrow[\mathcal{L}^{-1}]{}$	Time domain (t)
System function $G(s) = \dfrac{Q(s)}{P(s)}$	$\longleftrightarrow$	weight function $w(t)$
Response $X(s) = G(s)F(s)$	$\longleftrightarrow$	Response $x(t) = (w * f)(t)$

Look at the bottom row of the table: it shows a formula for $X(s)$ and one for $x(t)$. Thus, the Laplace transform of the time domain formula must equal the frequency domain formula. We state this as a theorem

Theorem: For any two functions $f(t)$ and $g(t)$ with Laplace transforms $F(s)$ and $G(s)$ we have

$$\mathcal{L}(f * g) = F(s) \cdot G(s). \qquad (11)$$

$$\mathcal{L}(f * g; s) = G(s)F(s)$$

This is a great new entry for our table: Laplace turns convolution into multiplication.

Comparing equations (**??**) and (**??**) we see that

$$\mathcal{L}(w * f) = W(s) \cdot F(s). \qquad (12)$$

It appears that Laplace transforms convolution into multiplication. Technically, we only proved equation (12) when one of the functions is the weight function, but the formula holds in general.

Remarks:
1. This theorem gives us a way to prove that convolution is commutative. It is just the commutivity of regular multiplication on the s-side.

$$\mathcal{L}(f * g) = F \cdot G = G \cdot F = \mathcal{L}(g * f).$$

2. We could also prove the commutivity of convolution by writing down the appropriate double integrals and changing the order of integration.