Frequency response; frequency domain; Bode and Nyquist plots; transfer functions
Class 2, 18.031
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1 Introduction

We will start by review the notions of gain, phase lag and frequency response from 18.03. We then introduce the Nyquist and Bode plots which are graphs that represent the frequency response. We will discuss the transfer of system function which will extend the notion of frequency response to include *complex frequencies*. Finally we will introduce block diagrams, which give us a simple and useful way of presenting systems.

2 Complex gain, gain and phase lag

In the notes for class 1 we reviewed these concepts. We summarize them here so we can reference them in what follows. Everything in this section is important. If some of it is hazy you should go back and review the details in the reading for class 1.

Our basic LTI sytem with input $f(t)$ satisfies the differential equation

$$P(D)x = Q(D)f(t). \tag{1}$$

The exponential response formula solves Equation 1 in the case $f(t)$ is an exponential function.

Theorem. (ERF) If $P(s) \neq 0$ then the equation

$$P(D)x = Q(D)e^{st} \tag{2}$$

has a particular solution

$$x_p = \frac{Q(s)}{P(s)}e^{st}.$$

The ERF is remains valid if s is complex. We used this to produce the sinusoidal response formula.

Theorem. (SRF) If $P(i\omega) \neq 0$ then the equation

$$P(D)x = Q(D)F_0 \cos(\omega t) \tag{3}$$

has a particular solution

$$x_p = A\cos(\omega t - \phi), \quad \text{where} \quad A = F_0|Q(i\omega)/P(i\omega)|, \text{ and } \phi = -\text{Arg}(Q(i\omega)/P(i\omega)).$$

and the complex replacement equation

$$P(D)z_p = Q(D)F_0 e^{i\omega t} \tag{4}$$

has a particular solution

$$z_p = F_0 \frac{Q(i\omega)}{P(i\omega)} e^{i\omega t}.$$

We had names for all the pieces in the SRF.

1. In equation 3 we consider $F_0 \cos(\omega t)$ to be the input.

2. F_0 is the amplitude of the input. It has the same units as the input.

3. ω is the angular frequency of the input. It has units of radians/time.

4. $A = F_0 |Q(i\omega)/P(i\omega)|$ is the amplitude of the response. It has the same units as the response quantity.

5. $g(\omega) = |Q(i\omega)/P(i\omega)|$ is the gain. The gain is the factor that the input amplitude is multiplied by to get the amplitude of the response. It has the units needed to convert input units to output units.

6. $\phi = -\text{Arg}(Q(i\omega)/P(i\omega))$ is the phase lag. It has units of radians.

7. ϕ/ω is the time lag. This has units of time.

8. $Q(i\omega)/P(i\omega)$ is the complex gain.

It is important to remember that gain and phase are functions of the input frequency ω.

3 Frequency response

The sinusoidal solution to Equation 3 is completely determined by the gain and phase lag. That is, if the gain is g and the phase lag is ϕ then the sinusoidal solution is

$$x_p = g F_0 \cos(\omega t - \phi).$$

Now, both the gain and phase lag are functions of ω. Considered together the gain and phase lag are called the frequency response of the system. You should take this to mean they determine how the system responds to a given input frequency.

We reiterate their formulas:

$$g(\omega) = |Q(i\omega)/P(i\omega)| \text{ and } \phi(\omega) = -\text{Arg}(Q(i\omega)/P(i\omega)).$$

Taken separately we call $g(\omega)$ the amplitude response of the system and $\phi(\omega)$ is the phase response of the system.

3.1 Amplitude response and practical resonance

The gain $g(\omega)$ is a function of ω. It tells us the size of the system's response to the given input frequency. We can graph $g(\omega)$ vs ω and if the amplitude has a peak at ω_r we call this the practical resonance frequency.

Example 1. Consider the system

$$m\ddot{x} + b\dot{y} + kx = kF_0\cos(\omega t)$$

where we consider $F_0\cos(\omega t)$ to be the input. Graph its gain function and find the practical resonant frequency if it exists.

<u>answer:</u> The sinusoidal solution to this equation is

$$x_p(t) = \frac{kF_0\cos(\omega t - \phi)}{|P(i\omega)|} = \frac{kF_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}\cos(\omega t - \phi) \qquad (\text{where } \phi = \text{Arg}(P(i\omega))).$$

The gain is

$$g = g(\omega) = \frac{k}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

Here we consider the system parameters m, b, k to be fixed and that the gain depends on the input parameter ω.

Finding practical resonance: If it exists the practical resonance frequency ω_r is the frequency where where $g(w)$ has a maximum. Now, $g(\omega)$ has a maximimum where the expression under the square root in the denominator has a minimum. That is, where
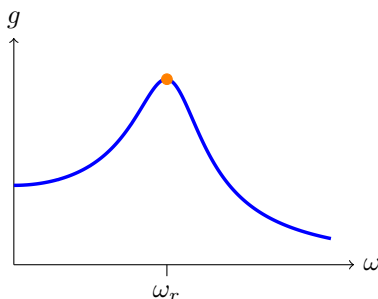
$$f(\omega) = (k - m\omega^2)^2 + b^2\omega^2$$

has a minimum. Taking the derivative and setting it to 0 we get

$$f'(\omega) = -4m\omega(k - m\omega^2) + 2b^2\omega = 0.$$
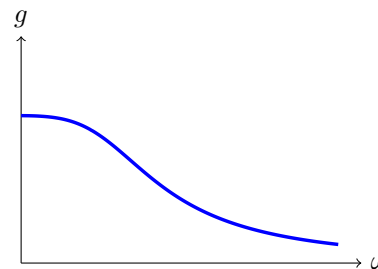
Solving for ω we find

$$\omega_r = 0 \text{ or } \omega_r = \sqrt{k/m - b^2/2m^2} = \sqrt{\omega_0^2 - b^2/2m^2}.$$

The last expression is given because it expresses ω_r in terms of the *natural frequency* $\omega_0 = \sqrt{k/m}$.

$\omega_r = \sqrt{\omega_0^2 - b^2/2m^2}$
(**practical resonance**).

(no practical resonance).

Notice in this example that if the damping gets too large there is no practical resonance.

This example is illustrated in the MIT Mathlet *Amplitude Phase Second Order I*. (See
http://mathlets.org/mathlets/amplitude-and-phase-2nd-order/

3.2   The simple harmonic oscillator

The simple harmonic oscillator is modeled by the equation

$$m\ddot{x} + kx = f(t).$$

Example 2. Find the full solution to the DE: $m\ddot{x} + kx = F_0 \cos \omega t$.

<u>answer:</u> We do not show all the algebra. You should try it for yourself.

Homogeneous solution:
$$x_h = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$
where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator.

Particular solution:

$$x_p = \begin{cases} \dfrac{F_0 \cos \omega t}{|k - m\omega^2|} = \dfrac{F_0 \cos \omega t}{m|\omega_0^2 - \omega^2|} & \text{if } \omega < \omega_0 \\[3mm] \dfrac{F_0 \cos(\omega t - \pi)}{|k - m\omega^2|} = \dfrac{-F_0 \cos \omega t}{m|\omega_0^2 - \omega^2|} & \text{if } \omega > \omega_0 \\[3mm] \dfrac{F_0\, t \cos(\omega_0 t - \pi/2)}{2m\omega_0} = \dfrac{F_0\, t \sin(\omega_0 t)}{2m\omega_0} & \text{if } \omega = \omega_0. \end{cases}$$

The bottom case ($\omega = \omega_0$) requires complex replacement and the extended ERF.

The general solution (to the inhomogeneous DE) is given by $x = x_p + x_h$.

Resonance and amplitude response of the simple harmonic oscillator

The solution for x_p shows that the gain of the system is given by

$$g = g(\omega) = \frac{1}{m|\omega_0^2 - \omega^2|}.$$

This is a function of ω and the right hand plot below shows its graph.

Note, it is similar to the damped amplitude response except the peak is infinitely high. As w gets closer to ω_0 the amplitude increases without bound.

When $\omega = \omega_0$ we have $x_p = \frac{F_0\, t \sin \omega_0 t}{2m\omega_0}$. This is called pure resonance. The frequency ω_0 is called the resonant or natural frequency of the system. The left-hand plot below graphs x_p in this case. Notice how the response is oscillatory but not periodic. The factor of t in x_p causes the amplitude to keep growing over time.

Note carefully the different units and different meanings in the plots below. The left hand plot is output vs. time (for a fixed input frequency) and the right hand plot is output amplitude vs. input frequency. Also, x and g are in physical units dependent on the system; t is in time; ω is in radians per unit time, e.g. radians/sec.
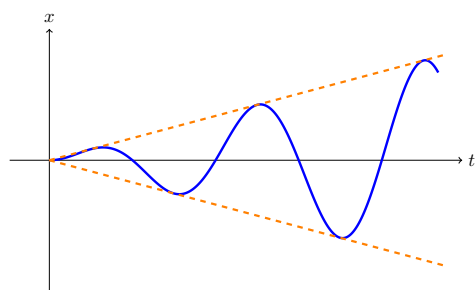
Resonant response $(\omega = \omega_0)$
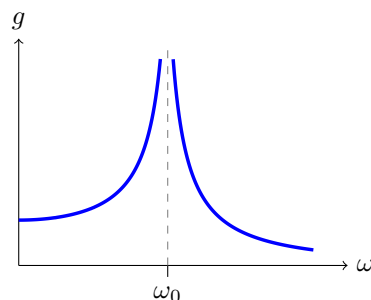
Undamped amplitude response

4   Bode and Nyquist plots

In brief, Bode (rhymes with roadie) plots show the the frequency response of a system. There are two Bode plots one for gain (or magnitude) and one for phase. The amplitude response curves given above are examples of the Bode gain plot.

The Nyquist plot combines gain and phase into one plot in the complex plane. It is drawn by plotting the complex gain $g(iw)$ for all frequencies w. That is, the plot is a curve in the plane parametrized by ω.

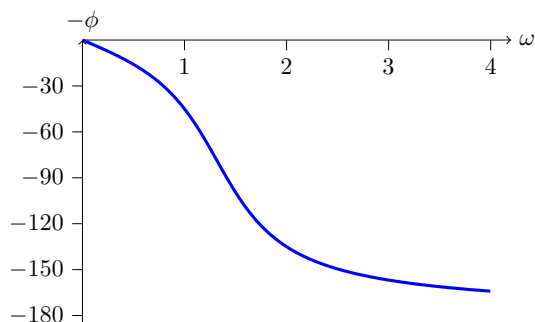**Example 3.** Give the Bode and Nyquis plots for the system $\ddot{x} + \dot{x} + 2x = 2f(t)$ with input $f(t)$.

answer: The system has complex gain $2/P(iw)$, where $P(s) = s^2 + s + 2$. So the gain and phase are

$$g(\omega) = \frac{2}{|P(i\omega)|} = \frac{2}{|1 - \omega^2 + 2i\omega|} = \frac{2}{\sqrt{(1 - \omega^2)^2 + (2\omega)^2}} \qquad -\phi = \frac{2}{P(i\omega)} = -\mathrm{Arg}(P(i\omega)) = \mathrm{Arg}(1 - \omega^2 + 2i\omega).$$
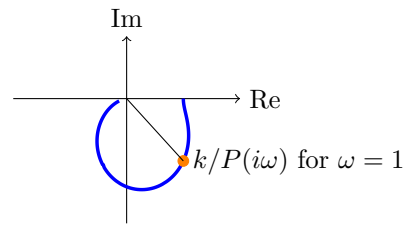
Note that we write $-\phi$ because we are thinking of ϕ as the phase lag, so $-\phi$ is the phase. The Bode and Nyquist plots are shown below.
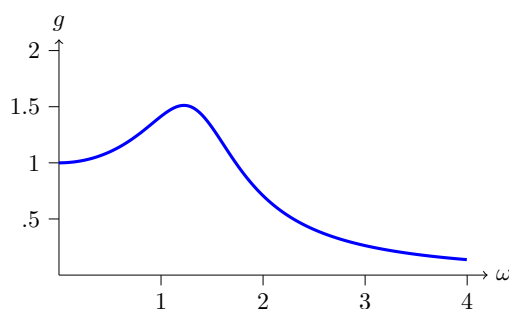

Bode gain plot

Bode phase plot (in degrees)

Nyquist plot

The best way to get a feel for these plots is to play with the MIT Mathlet *Amplitude Phase Second Order I*. (See `http://mathlets.org/mathlets/amplitude-and-phase-2nd-order/` Especially watch how the value of the gain and phase change as you move the ω slider. Also watch how the gain and phase are reflected in the x vs t plots of input and output.

A more sophisticated mathlet that you will understand once we talk about zeros and poles is called *Bode and Nyquist Plots*. You can see it at `http://mathlets.org/mathlets/bode-and-nyquist-plots/`

In this class we will usually draw Bode plots on a linear-linear scale. In other places you are likely to see the gain plot drawn on a log-log scale and the phase plot drawn on a linear-log scale.

Here are some examples using log scales taken from `https://en.wikipedia.org/wiki/Bode_plot?sa=X&ved=0ahUKEwjKwcPpvaPKAhVIqx4KHZN3C3MQ9QEIGTAA#/media/File:Bode_High-Pass.PNG` Notice that the vertical axis is in units of decibels, this is a log scale computed with the formula $20 * \log_{10}(\text{gain})$
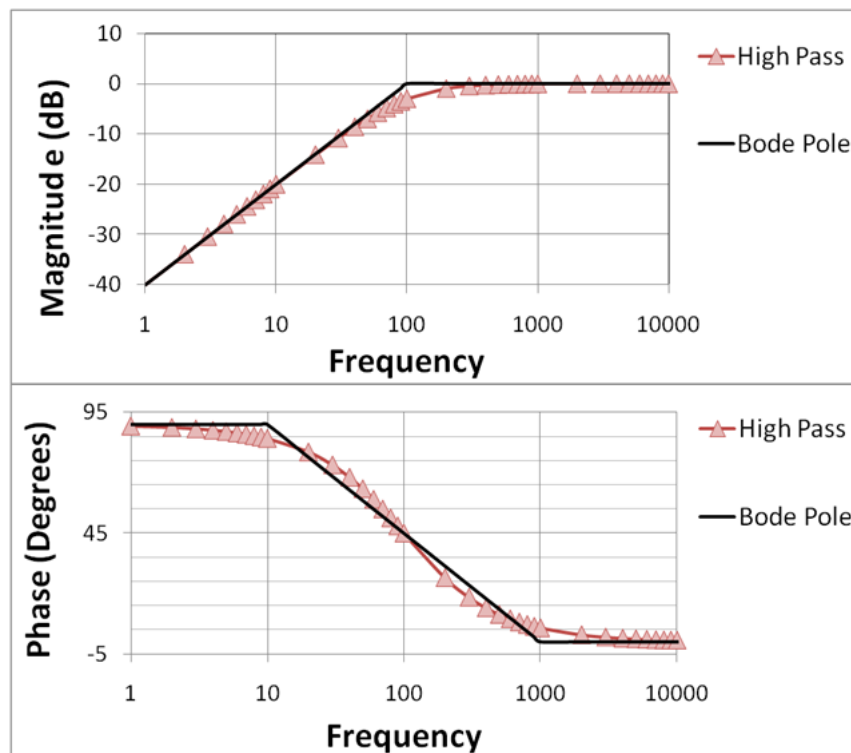
5 Transfer function

Here we introduce the first term you probably did not see in 18.03. Recall the system in Equation 1 and the theorem just below it. For your reading pleasure we reiterate them here. The system is

$$P(D)x = Q(D)f(t), \tag{5}$$

where $f(t)$ is considered the input.

Theorem. (ERF) If $P(s) \neq 0$ then the equation

$$P(D)x = Q(D)\mathrm{e}^{st} \tag{6}$$

has a particular solution

$$x_p = \frac{Q(s)}{P(s)}\mathrm{e}^{st},$$

where, as we have seen, the variable s is allowed to be complex.

Definition. As a function of the complex variable s we call the function

$$H(s) = \frac{Q(s)}{P(s)}$$

the transfer function of the system in Equation 5.

You can think of the name as meaning that it transfers the input e^{st} to the response $x_p = H(s)\mathrm{e}^{st}$. The transfer function is also often called the system function.

We have already spent a fair amount of time with this function, though much of it was specialized to the case $s = i\omega$. Let's go over what we know.

1. If the system has input $\cos(\omega t)$ then the complexified system has input $\mathrm{e}^{i\omega t}$ and
   (i) $H(i\omega) = Q(i\omega)/P(i\omega)$ is the complex gain
   (ii) $|H(i\omega)| = |Q(i\omega)/P(i\omega)|$ is the gain
   (iii) $-\mathrm{Arg}(H(i\omega)) = -\mathrm{Arg}(Q(i\omega)/P(i\omega))$ is the phase lag.

2. s has units of frequency, that is 1/time.

Because of these properties s is called the complex frequency. By letting $s = i\omega$ the transfer function $H(s) = H(i\omega)$ determines the entire frequency response. This helps to explain its importance: by widening our view to all complex s we can get a better view of the frequency response which is our true interest.

6 Block diagrams and coupled systems

There is a convenient way to represent LTI systems with block diagrams. The system is represented as a box, usually with the transfer function written inside it. The input and outputs are represented by arrows. We illustrate with a few examples.

We have seen the the system $P(D)y = Q(D)f$ (with input f) has transfer function $H(s) = Q(s)/P(s)$. We can represent this with a **block diagram**.
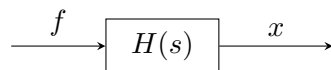
You should read this diagram as f goes into the system, is processed and x comes out. If the input $f(t) = e^{st}$ then we take the particular solution $x_p = H(s)e^{st}$ for the output.

Now consider the *cascaded* system

$$P_1(D)x = f, \qquad P_2(D)y = x.$$

The input to the cascade is f, the first equation uses f as input and outputs x, which is then the input to the second equation which outputs y. Let H_1 be the transfer function for the first equation and H_2 that for the second.

Now, if $f(t) = e^{st}$ then by the ERF $x(t) = H_1(s)e^{st}$ and again by the ERF $y(t) = H_2(s)H_1(s)e^{st}$. This shows that the entire cascade has transfer function $H_2(s) \cdot H_1(s)$, i.e. in cascaded systems the transfer functions multiply.
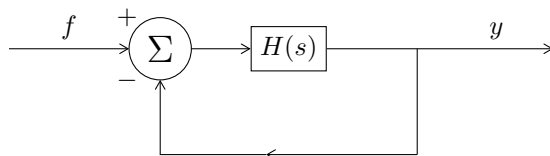
As a *block diagram* we have
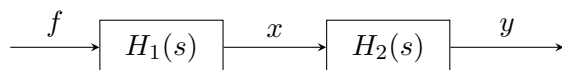
The equivalent system with transfer function $H_1(s)H_2(s)$ can be diagrammed as
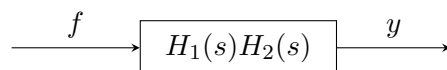
Block Diagrams with Closed Loops

Here's a more complicated system with a feedback loop, so that the output y is also fed back through the system.

The symbol Σ indicates how to combine the inputs, i.e., $v = f - y$.

We will really learn how to do the algebra that determines the transfer function once we have learned about the Laplace transform. You can try to discover it by letting $f(t) = e^{st}$, $v = ae^{st}$ and $y = be^{st}$. The end result will be Black's formula for the transfer function of the system with input f and output y:

$$\text{transfer function} = \frac{H(s)}{1 + H(s)}.$$