Constant coefficient linear ODEs: review from 18.03
Class 1, 18.031
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1 Prerequisites: complex arithmetic and exponentials

A course equivalent to 18.03 is a prerequisite for 18.031. As in 18.03 we will make frequent use of complex numbers. In particular, we will use complex exponentials and Euler's formula. We will not use class time to review complex numbers, but if you need to review them, we've provided a thorough review (courtesy of 18.03) in the review section on the course website.

2 Introduction

The main object of study in 18.031 will be linear time invariant (LTI) systems. An important class of LTI systems are modeled by the constant coefficient linear differential equations you studied in 18.03.

This note gives a reasonably complete review of the 18.03 material on linear constant coefficient differential equations. We will review the basic definitions and notations; solving homogeneous and inhomogeneous equation; the Exponential Response Formula and the Sinusoidal Response Formula; stability and frequency response. At the end we briefly introduce two new ideas: the transfer function and block diagrams.

In 18.031 will use standard LTI terminology and call the zeros of $p(s)$ the poles of the transfer function. In later classes we will learn to summarize LTI systems in the pole diagram. Eventually we will work with LTI systems that are not modeled by differential equations, e.g. systems with feedback loops. We will see that these systems do not have characteristic polynomials, but they have transfer functions, poles and pole diagrams and their meaning will be essentially the same as for differential equations.

3 Constant coefficient differential equations

In this section we'll remind you of the main definitions, techniques and theorems for constant coefficient differential equations.

3.1 Definition

A first order constant coefficient linear differential equation has the form

$$a\frac{dx(t)}{dt} + bx(t) = f(t), \tag{1}$$

where the coefficients a and b (of $\dot{x}$ and x) are constants –hence the name constant coefficient. The right hand side $f(t)$ is allowed to be any function. We will usually abbreviate Equation

1 by

$$a\dot{x} + bx = f(t)$$

A second order constant coefficient differential equation has the form

$$m\ddot{x} + b\dot{x} + kx = f(t),$$

where the coefficients m, b, k are constants. In this equation we used the standard symbols, m, b, k for modeling spring-mass-damper systems. Of course, for other physical systems we will use other symbols.

The general n^{th} order constant coefficient differential equation has the form

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \ldots + a_1 \dot{x} + a_0 x = f(t),$$

where the coefficients a_j are constants.

Just for completeness we will offer some explicit examples of constant coefficients equations:
Example 1.

1. $\dot{x} + 5x = 0$ (first order)
2. $\dot{x} + 5x = \cos(3t)$ (first order)
3. $2\ddot{x} + 8\dot{x} + 7x = 0$ (second order)
4. $2\ddot{x} + 8\dot{x} + 7x = \mathrm{e}^{4t}$ (second order)
5. $2\ddot{z} + 8\dot{z} + 7z = \mathrm{e}^{i3t}$ (second order complex)
6. $3x^{(4)} + 4x^{(3)} + 5\ddot{x} + 6\dot{x} + 7x = 8.$ (fourth order).

When $f(t) = 0$ we call the equation homogeneous. Thus equations 1 and 3 in the previous example are homogeneous. When $f(t)$ is not 0, we call the differential equation inhomogeneous. Thus, equations 2, 4, 5, 6 are inhomogeneous differential equations.

3.2 Operator notation

Recall the differential operator $D = \dfrac{d}{dt}$. So we have several different ways of writing derivatives:

$$Dx = \frac{dx}{dt} = \dot{x} \quad \text{and} \quad D^2 x = \frac{d^2x}{dt^2} = \ddot{x}.$$

Likewise we can use D to write differential equations. For example, items 1 and 4 in Example 1 can be written as

1. $Dx + 5x = 0$ *or* $(D + 5I)x = 0$
4. $2D^2x + 8Dx + 7x = \mathrm{e}^{4t}$ *or* $(2D^2 + 8D + 7I)x = \mathrm{e}^{4t}.$

(Here I is the identity operator.)

With operator notation we can easily write a linear constant coefficient equation of any order: If $P(s)$ is a polynomial of degree n then

$$P(D)x = f(t) \tag{2}$$

is a constant coefficient differential equation of order n.

Note on informal notation. In item 1 just above we were careful to include the identity operator so that in the expression $D + 5I$ we were summing two operators. We will often allow ourselves to use the informal notation $(D + 5)$ to mean the same thing.

3.3 Homogeneous and inhomogeneous equations

If $f(t)$ on the right hand side of Equation 2 is 0 then the differential equation is called homogeneous. If $f(t)$ is not 0 (or if we do not know it) we will refer to Equation 2 as an inhomogeneous equation.

3.4 Linearity and superpostion

The most important property of the differential operator $P(D)$ is linearity. Recall that this means that for any constants c_1, c_2 and any functions x_1, x_2 we have

$$P(D)(c_1x_1 + c_2x_2) = c_1P(D)x_1 + c_2P(D)x_2. \tag{3}$$

Of course this definition extends to operators besides $P(D)$: we say an operator T on functions is linear if

$$T(c_1x_1 + c_2x_2) = c_1Tx_1 + c_2Tx_2, \tag{4}$$

for any constants c_1, c_2 and functions x_1, x_2.

You will have seen in 18.03 or 18.06 the set of differentiable functions forms a vector space and that $P(D)$ and T in Equations 3 and 4 are examples of linear operators or maps on a vector space.

Linearity leads directly to a set of superposition principles for linear constant coefficient DEs. Before stating these principles we remind you of the definition of linear combination. For constants c_1 and c_2 we call

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

a linear combination of the functions $x_1(t)$ and $x_2(t)$.

Superposition principle for linear homogeneous DEs. Suppose x_1 and x_2 are solutions to the homogeneous equation

$$P(D)x = 0.$$

Then so are all linear combinations $x = c_1x_1 + c_2x_2$ of x_1 and x_2.

Proof. The proofs for all our superposition principles will be the same. We check the purported solution by substituting it into the DE and then verifying that it satisfies the equation. Since $P(D)$ is linear we know that

$$P(D)x = P(D)(c_1x_1 + c_2x_2) = c_1P(D)x_1 + c_2P(D)x_2 = 0 + 0 = 0.$$

Thus $P(D)x = 0$ which shows $x(t)$ is indeed a solution of the DE.

Superposition principle for linear inhomogeneous DEs. Suppose x_p is a solution to the inhomogeneous equation

$$P(D)x = f(t).$$

and x_h is a solution to the associated homogeneous equation $P(D)x = 0$ then $x = x_p + x_h$ is also a solution to the inhomogeneous equation.

Proof. Since $P(D)$ is linear we have

$$P(D)x = P(D)(x_p + x_h) = P(D)x_p + P(D)x_h = f + 0 = f.$$

Superposition of inputs. Suppose x_1 is a solution to the inhomogeneous equation $P(D)x = f_1(t)$ and x_2 is a solution to $P(D)x = f_2(t)$ then $x = c_1x_1 + c_2x_2$ is a solution to the equation $P(D)x = c_1f_1 + c_2f_2$.

Proof. Since $P(D)$ is linear we have

$$P(D)x = P(D)(c_1x_1 + c_2x_2) = c_1P(D)x_1 + c_2P(D)x_2 = c_1f_1 + c_2f_2.$$

3.5 Systems modeled by constant coefficient linear DEs

One of the reasons for our interest in constant coefficient linear DEs is that they model many interesting physical systems. We present here several examples which you probably saw in 18.03. We will not go through the physical derivation of these models.

Example 2. Exponential decay. The radioactive matter in a body decays at a rate proportional to the amount present. If $x(t)$ is the mass of radioactive matter in a body then we have the model $\dot{x} = -kx$, where k is a rate constant with units of 1/time. We usually write this differential equation as

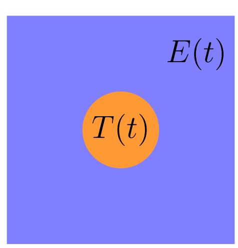
$$\dot{x} + kx = 0$$

If we are also adding radioactive matter to the body at the rate $f(t)$ (in units of mass/time) then the model becomes

$$\dot{x} + kx = f(t)$$

Example 3. Newton's law of cooling. Suppose two bodies at different temperatures are in contact. Newton's law says they will exchange heat at a rate proportional to the difference in their temperatures.

For example, suppose a heated body placed in an environment. Suppose also that the temperature of the body as a funtion of time is given by $T(t)$ and the temperature of the environment is given by $E(t)$.



Newton's law of cooling models this by the differential equation $\dot{T} = -k(T - E)$, where k is a physical constant depending on the materials and sizes of the bodies. The units for k are 1/time. We usually write the differential equation in the form

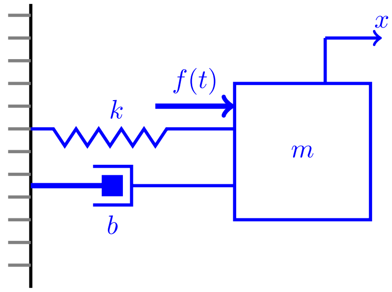
$$\dot{T} + kT = kE$$

This law assumes each body has a uniform temperature. This is clearly a simplification since each body will really have a temperature gradient across it. Nonetheless it is a useful first order approximation to reality. A more accurate model is given by the heat equation you also studied in 18.03.

Example 4. **Spring-mass-damper.** In the spring-mass-damper system shown below a mass is driven by a force pushing directly on it. The diplacement of the mass from equilibrium is modeled by

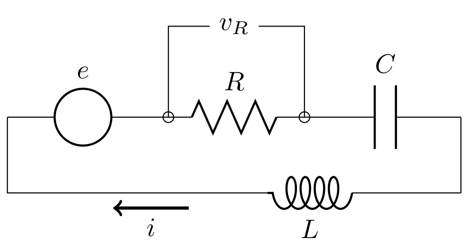
$$m\ddot{x} + b\dot{x} + kx = f(t),$$

where x is the displacement, m is the mass, b is the damping constant, and k is the spring constant. Compatible units are x in meters, m in kg, b in kg/sec, k in kg/sec^2, and f in Newtons.



We get different equations if the force is applied to the spring or the damper instead of the mass. We can get more complicated systems by coupling together two or more spring-mass systems.

Example 5. **RLC circuit.** Suppose a resistor R, capacitor C and inductor L are placed in series in a circuit driven by a voltage e.



The current i is modeled by the equation

$$Li'' + Ri' + \frac{1}{C}i = e'.$$

(Here we use primes instead of dots to indicate time derivatives, so we do not get confused by the dot over the i.)

For the same physical setup we might be interested in a different quantity. For example, the voltage v_R across the resistor is modeled by

$$L\ddot{v}_R + R\dot{v}_R + \frac{1}{C}v_r = \dot{e}$$

3.6 Solving homogeneous equations using characteristic roots

If Equation 2 has order n then it has n independent solutions. In the language of linear algebra, we can say the space of solutions is n-dimensional. In 18.03 you learned to solve these equations using the roots of the characteristic polynomial. We will recall the methods and their justification with a series of examples.

Example 6. (Long form solution with justification)

Solve the equation $\ddot{x} + 8\dot{x} + 7x = 0$.

answer: We use the method of optimism to guess a solution of the form $x(t) = \mathrm{e}^{st}$. We then substitute our guess into the equation, which gives

$$\ddot{x} + 8\dot{x} + 7x = s^2\mathrm{e}^{st} + 8s\mathrm{e}^{st} + 7\mathrm{e}^{st} = 0.$$

Since e^{st} is never 0 it is okay to divide it out to get the characteristic equation

$$s^2 + 8s + 7 = 0.$$

The roots of this equation are $s = -1, -7$. So the method of optimism has given us two modal solutions:

$$x_1(t) = \mathrm{e}^{-t} \quad \text{and} \quad x_2 = \mathrm{e}^{-7t}.$$

The superposition principle for homogeneous equations then gives us that the general solution to the DE is

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1\mathrm{e}^{-t} + c_2\mathrm{e}^{-7t}.$$

Example 7. (Model solution) Solve $\ddot{x} + 4\dot{x} + 3x = 0$.

answer: Characteristic equation: $s^2 + 4s + 3 = 0$.

Roots: $s = -1, -3$.

Two modal solutions: $x_1(t) = \mathrm{e}^{-3t}$, $x_2(t) = \mathrm{e}^{-t}$.

General solution by superposition: $x(t) = c_1x_1 + c_2x_2 = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t}$.

Complex roots. Amazingly, superposition makes complex roots easy to handle.

Theorem. Consider the homogeneous equation $m\ddot{x} + b\dot{x} + kx = 0$. If $z(t) = x_1(t) + ix_2(t)$ is a solution then so are x_1 and x_2.

(Here we are assuming m, b, k, $x_1(t)$ and $x_2(t)$ are all real.)

Proof. This is just another application of superposition. We first remind you that if a complex number $z = a + bi = 0$ then $a = b = 0$, i.e. both its real and imaginary parts are 0. Now we can give the proof proper:

By assumption z is a solution to the DE so

$$m\ddot{z} + b\dot{z} + kz = 0.$$

Replacing z by $x_1 + ix_2$ gives

$$m(\ddot{x}_1 + i\ddot{x}_2) + b(\dot{x}_1 + i\dot{x}_2) + k(x_1 + ix_2) = 0.$$

Therefore

$$(m\ddot{x}_1 + b\dot{x}_1 + kx_1) + i(m\ddot{x}_2 + b\dot{x}_2 + kx_2) = 0.$$

The bottom equation shows that both the real and imaginary parts must be 0, i.e. $m\ddot{x}_1 + b\dot{x}_1 + kx_1 = 0$ and $m\ddot{x}_2 + b\dot{x}_2 + kx_2 = 0$. This says exactly that x_1 and x_2 are solutions to the DE, which is what we needed to prove.

We can use the theorem to find the real solution when we have complex roots.

Example 8. (Long form solution) Solve $\ddot{x} + 2\dot{x} + 4x = 0$.

answer: The characteristic equation is $s^2 + 2s + 4 = 0$. This has roots $s = -1 \pm \sqrt{3}\,i$. Thus we have two (complex) exponential solutions

$$z_1(t) = \mathrm{e}^{(-1+\sqrt{3}\,i)t}, \quad z_2(t) = \mathrm{e}^{(-1-\sqrt{3}\,i)t}$$

Using Euler's theorem we can write these equations as

$$z_1 = \mathrm{e}^{-t}\cos(\sqrt{3}t) + i\mathrm{e}^{-t}\sin(\sqrt{3}t), \quad z_2 = \mathrm{e}^{-t}\cos(\sqrt{3}t) - i\mathrm{e}^{-t}\sin(\sqrt{3}t),$$

Except for the minus sign, z_1 and z_2 have the same real and imaginary parts. The theorem says that these parts are also solutions to the differential equation. Thus we have two independent solutions

$$x_1(t) = \mathrm{e}^{-t}\cos(\sqrt{3}t), \quad x_2(t) = \mathrm{e}^{-t}\sin(\sqrt{3}t).$$

N.B. $x_2(t)$ is real, it does not include the factor of i. The superposition principle now gives us the general solution to the DE is

$$x(t) = c_1\mathrm{e}^{-t}\cos(\sqrt{3}t) + c_2\mathrm{e}^{-t}\sin(\sqrt{3}t).$$

Example 9. (Model solution) Solve $\ddot{x} + 2\dot{x} + 4x = 0$.

answer: Characteristic equation: $s^2 + 2s + 4 = 0$.
Roots: $s = (-2 \pm \sqrt{4-16})/2 = -1 \pm \sqrt{3}\,i$.
Two solutions: $x_1(t) = \mathrm{e}^{-t}\cos(\sqrt{3}t)$, $x_2(t) = \mathrm{e}^{-t}\sin\sqrt{(3t)}$.
General *real valued* solution:

$$x(t) = c_1x_1(t) + c_2x_2(t) = c_1\mathrm{e}^{-t}\cos(\sqrt{3}t) + c_2\mathrm{e}^{-t}\sin(\sqrt{3}t)$$

The DE for the has the form

$$m\ddot{x} + kx = 0$$

Example 10. Solve (unforced simple harmonic oscillator) $2\ddot{x} + 8x = 0$.

answer: This is the equation for the unforced simple harmonic oscillator $m\ddot{x} + kx = 0$, with $m = 2$ and $k = 8$.

Char. eq: $2s^2 + 8 = 0$.
Roots: $s = \pm 2i$.
Complex solutions: $z_1(t) = e^{2it}$, $z_2(t) = e^{-2it}$.
General *real valued* solution: $x = c_1 \cos(2t) + c_2 \sin(2t)$.

Example 11. Suppose the equation $P(D)x = 0$ is fifth order and has characteristic roots -2, $1 \pm 7i$, $\pm 3i$. What is the general real valued solution?

answer: All we need is the roots to find the solution:

$$x(t) = c_1 e^{-2t} + c_2 e^t \cos(7t) + c_3 e^t \sin(7t) + c_4 \cos(3t) + c_5 \sin(3t).$$

Example 12. Repeated roots. Suppose the characteristic roots of $p(D)x = 0$ are 3, 3, 5, 5, 5, 2. Find the general solution to this DE.

answer: Recall that with repeated roots we bring in factors of t to find the extra modal solutions. So the general solution is

$$x(t) = c_1 e^{3t} + c_2 t e^{3t} + c_3 e^{5t} + c_4 t e^{5t} + c_5 t^2 e^{5t} + c_6 e^{2t}.$$

Example 13. Suppose the characteristic roots of $P(D)x = 0$ are

$$1 + 2i,\ 1 + 2i,\ 1 - 2i,\ 1 - 2i,\ -3.$$

The general real valued solution is

$$x = c_1 e^t \cos(2t) + c_2 e^t \sin(2t) + c_3 t e^t \cos(2t) + c_4 t e^t \sin(2t) + c_5 e^{-3t}.$$

3.7 Systems, signals, input and response

For physical systems we often have a notion of what we consider the input to the system and what is the response.

Example 14. If have a spring-mass system and I apply a time varying force to the mass it is reasonable to consider the force, $f(t)$ as the input and the position $x(t)$ of the mass as the output. Assume also that the system is damped proportional to the velocity. Newton's second law says that

$$m\ddot{x} = -kx + b\dot{x} + f(t).$$

The systems and signals way of writing this is to have everything to do with the output on the left and everything to do with the input on the right. So the equation becomes.

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

The terminology we use is that $f(t)$ is the input signal, $x(t)$ is the response or output signal and the left hand side represents the system.

We cannot emphasize too quickly or too strongly that the left hand side is not necessarily the input, but it is always related to the input. Two examples will get our point across.

Example 15. (Newton's law of cooling.) Imagine you are manufacturing glass. After being heated and shaped the glass has to be cooled to room temperature in a controlled manner. This process is called annealing and the annealing schedule gives the temperature of the glass as a function of time. To achieve this schedule the temperature of the cooling environment has to be set. We will model this using Newton's law of cooling. Let $T(t)$ be the temperature of the glass and $E(t)$ the temperature of the environment. As in Example 3, this is modeled by

$$\dot{T}(t) + kT(t) = kE(t).$$

In this case you will want to know what $E(t)$ should be, i.e you will consider $E(t)$ to be the input signal and $T(t)$ to be the output. Notice that the right hand side of the equation is $kE(t)$ which not the input, but it is derived from the input.

Example 16. (RLC circuits.) In Example 5 we modeled an RLC circuit by

$$Li'' + Ri' + \frac{1}{C}i = e'.$$

If you imagine the unpowered device as the system and you run it by connecting it to a power source then it's reasonable to think of the voltage e as the input and the current i as the ouptut. Again the right hand side is not the input, but is derived from it.

3.8 More general systems

In 18.031 we will want to look at systems of the form

$$P(D)x = Q(D)f,$$

where P and Q are polynomials, f is the input and x is the response.

3.9 Solving inhomogenous linear equations

The superposition principle makes solving inhomogeneous linear equations conceptually very easy.

Theorem. The general solution to the equation

$$P(D)x = Q(D)f \tag{5}$$

is given by

$$x(t) = x_p(t) + x_h(t),$$

where x_p is *any* one solution to Equation 5 and x_h is the general solution to the homogeneous equation $P(D)x = 0$. We call x_p a particular solution to the DE.

The theorem reduces solving the equation to finding one particular solution and then finding the general homogeneous equation using the characteristic roots. Of course finding a particular solution might be difficult, but for the cases we care about we have good methods which are shown in the subsections below.

3.10 Exponential input and the Exponential Response Formula

Theorem. (Exponential Response Formula (ERF))

The equation
$$P(D)x = \mathrm{e}^{st}$$
has a particular solution
$$x_p(t) = \begin{cases} \dfrac{\mathrm{e}^{st}}{P(s)} & \text{if } P(s) \neq 0 \\ \dfrac{t\mathrm{e}^{st}}{P'(s)} & \text{if } P(s) = 0 \text{ and } P'(s) \neq 0 \\ \dfrac{t^2\mathrm{e}^{st}}{P''(s)} & \text{if } P(s) = 0,\ P'(s) = 0 \text{ and } P''(s)! = 0. \end{cases}$$

The pattern continues if $P(s) = P'(s) = P''(s) = 0$ etc. Typically we call the cases where $P(s) = 0$ the extended ERF.

Proof: Using the method of optimism, try a solution of the form $x = c\mathrm{e}^{st}$. Substitution gives
$$P(D)x = P(s)c\mathrm{e}^{st} = \mathrm{e}^{st}.$$
Thus if $P(s) \neq 0$, we can take $c = 1/P(s)$ and we have the solution $x_P = \mathrm{e}^{st}/P(s)$ given in the theorem.

For the cases where $P(s) = 0$ note that we have
$$P(D)\mathrm{e}^{st} = P(s)\mathrm{e}^{st} \tag{6}$$
We differentiate (6) *with respect to* s. This gives
$$P(D)(s\mathrm{e}^{st}) = P'(s)\mathrm{e}^{st} + P(s)t\mathrm{e}^{st}.$$
Since $P(s) = 0$, the second term on the left is 0 and we have $P(D)(t\mathrm{e}^{st}) = P'(s)\mathrm{e}^{st}$. Dividing by $P'(s)$ proves the theorem in the case $P(s) = 0$ and $P'(s) \neq 0$.

We can continue in this manner if $P(s) = P'(s) = 0$ etc.

Example 17. Let $P(D) = D^2 + 4D + 5I$. Find a particular solution to $P(D)x = \mathrm{e}^{-t}$.

<u>**answer:**</u> We simply apply the Exponential Response Formula (ERF): $P(-1) = 2$, thus $x_p = \mathrm{e}^{-t}/2$ is a particular solution.

3.11 Sinusoidal input and the Sinusoidal Response Formula

We will first work an example the shows how to handle sinusoidal input using complex replacement.

Example 18. Let $P(s) = s^2 + 4s + 5$ and find a particular solution to $P(D)x = \cos(2t)$.

<u>**answer:**</u> (Long form of the solution)

Define y as a solution to the equation $P(D)y = \sin(2t)$, and let $z = x + iy$. We have
$$P(D)z = P(D)x + iP(D)y = \cos(2t) + i\sin(2t) = \mathrm{e}^{2it}.$$

This technique of replacing x by z and $\cos(2t)$ by e^{2it} is called **complex replacement** or **complexification**.

Now nothing in the ERF prohibits s from being complex. Since our new equation has an exponential on the right hand side the ERF gives us the solution to the complexified equation:

$$z_p(t) = \frac{e^{2it}}{p(2i)} = \frac{e^{2it}}{1+8i}.$$

Since $z = x + iy$ we have $x = \text{Re}(z)$. Thus, we have a solution to the original:

$$x_p = \text{Re}(z_p) = \text{Re}\left(\frac{e^{2it}}{1+8i}\right).$$

The best way to find and display x_p is in amplitude-phase form. This is done as follows:

We can write $P(2i) = 1 + 8i$ in polar form:

$$P(2i) = 1 + 8i = |P(2i)|e^{i\phi}, \text{ where } |P(2i)| = \sqrt{65} \text{ and } \boxed{\phi = \text{Arg}(P(2i)) = \tan^{-1}(8)}.$$

Thus

$$z_p = \frac{e^{2it}}{P(2i)} = \frac{e^{2it}}{\sqrt{65}e^{i\phi}} = \frac{1}{\sqrt{65}}e^{(2t-\phi)i}$$

In polar form taking the real part is easy:

$$x_p = \text{Re}(z_p) = \frac{1}{\sqrt{65}}\cos(2t - \phi).$$

Let's do a few more variations on this theme.

Example 19. Let $P(s) = s^2 + 4s + 5$ and find a solution to $P(D)x = e^{-t}\cos 2t$.

answer: (Short form of solution.)

Complex replacement:

$$P(D)z = e^{-t}e^{2ti} = e^{(-1+2i)t}, \quad x = \text{Re}(z).$$

Side work (put $P(-1+2i)$ in polar form):

$$P(-1+2i) = -2 + 4i = 2\sqrt{5}e^{i\phi},$$

where $\phi = \text{Arg}(-2+4i)$. Since $-2+4i$ is in the second quadrant we know that $\phi = \tan^{-1}(-2)$ is between $\pi/2$ and π. We abbreviate this by saying that ϕ is in the second quadrant.

ERF and putting z_p in polar form:

$$z_p(t) = \frac{e^{(-1+2i)t}}{P(-1+2i)} = \frac{e^{(-1+2i)t}}{-2+4i} = e^{-t}\frac{e^{i(2t-\phi)i}}{2\sqrt{5}}$$

Decomplexify:

$$x_p = \text{Re}(z_p) = \frac{e^{-t}}{2\sqrt{5}}\cos(2t - \phi).$$

We have found a particular solution in amplitude-phase form.

Example 20. (Extended ERF) Let $P(s)$ be as in the previous example and find a solution to $P(D)x = \mathrm{e}^{-2t}\cos t$.

Complexification:

$$P(D)z = \mathrm{e}^{-2t}\mathrm{e}^{ti} = \mathrm{e}^{(-2+i)t}, \quad x = \mathrm{Re}(z).$$

Side work: $P(-2+i) = 0$, so we need $P'(-2+i)$ in polar form:

$$P'(s) = 2s + 4, \text{ so }, P'(-2+i) = 2i = 2\mathrm{e}^{i\pi/2}.$$

Extended ERF:

$$z_p(t) = \frac{t\mathrm{e}^{(-2+i)t}}{P'(-2+i)} = t\mathrm{e}^{-2t}\frac{\mathrm{e}^{it}}{2\mathrm{e}^{i\pi/2}} = \frac{t\mathrm{e}^{-2t}}{2}\mathrm{e}^{i(t-\pi/2)}.$$

Decomplexify:

$$x_p = \mathrm{Re}(z_p) = \frac{t\mathrm{e}^{-2t}}{2}\cos(t - \pi/2)$$

(In this case, since $\cos(t-\pi/2) = \sin(t)$ we can also write $x_p = \frac{t\mathrm{e}^{-2t}}{2}\sin(t)$.)

Recap of the abstraction for sinusoidal input

To solve $P(D)x = \cos(\omega t)$:

1. Use complex replacement: $P(D)z = \mathrm{e}^{i\omega t}$, where $x = \mathrm{Re}(z)$.
2. Write $P(i\omega)$ in polar coordinates: $P(i\omega) = |P(i\omega)|\mathrm{e}^{i\phi}$, where $\phi = \mathrm{Arg}(P(i\omega))$.
3. Use the ERF:
$$z_p = \frac{\mathrm{e}^{i\omega t}}{P(i\omega)} = \frac{1}{|P(i\omega)|}\mathrm{e}^{i(\omega t - \phi)}.$$
4. Decomplexify: $x_p = \dfrac{1}{|P(i\omega)|}\cos(\omega t - \phi)$.

We use this so often that we record it as a theorem:

Theorem. (Sinusoidal Response Formula (SRF))

If $P(i\omega) \neq 0$ then a periodic particular solution to $P(D)x = \cos(\omega t)$ is given by

$$x_p = \frac{1}{|P(i\omega)|}\cos(\omega t - \phi), \text{ where } \phi = \mathrm{Arg}(P(i\omega)).$$

If $P(i\omega) = 0$ then we can find a particular solution by using complex replacement and the extended ERF.

4 Linear time invariant systems

Mathematically we say that a system is time invariant if whenever the input $f(t)$ has response $x(t)$ then for any constant a the input $f(t-a)$ has response $x(t-a)$.

Therefore the system

$$P(D)x = Q(D)f(t) \text{ is time invariant}$$

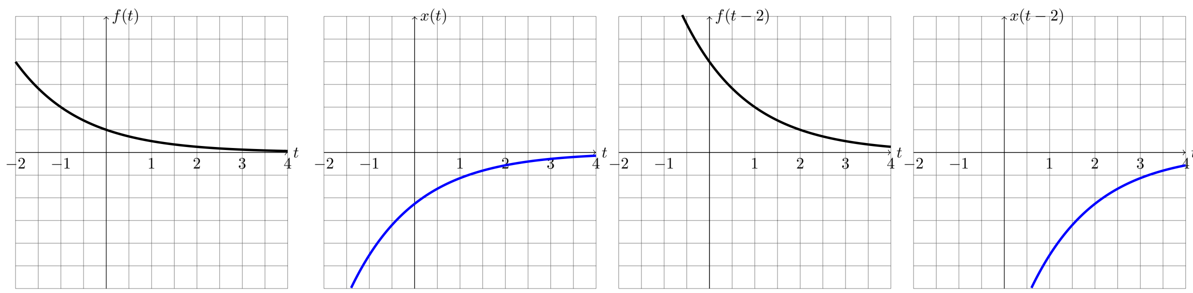
because for any constant a

$$\text{if } P(D)x(t) = Q(D)f(t) \text{ then } P(D)x(t-a) = Q(D)f(t-a).$$

Physically time invariance says that the system's response does not depend on what time the input begins. For example, if a spring-mass system is at equilibrium it will respond to a given force in the same way, no matter when the force is applied.

Example 21. You can easily check that $x(t) = -\mathrm{e}^{-0.7t}$ is a solution to $\dot{x} + 1.7x = \mathrm{e}^{-0.7t}$. It is also easy to check that the shifted function $x(t-2) = -\mathrm{e}^{-0.7(t-2)}$ is a solution to $\dot{x} + 1.7x = \mathrm{e}^{-0.7(t-2)}$. This is one example of time invariance.

The figure below shows this graphically. The pair of graphs on the left show the input $f(t)$ and response $x(t)$. The pair on the right shows the input $f(t-2)$ and response $x(t-2)$. Note the pair on the right are just the ones on the left shifted 2 units. If the times were not marked on the axis there would be no way to know which was which.



When the time invariant system is also linear we call it a linear time invariant system. We abbreviate this by calling it an LTI system.

Important note. So far we have only considered LTI systems that arise from linear constant coefficient DEs, but later we will see that there are other types. In particular, we will study LTI systems with feedback loops.

5 Stability

Definition: We will say that a system is stable if its long-term behavior doesn't depend (significantly) on initial conditions.

Example 22. Suppose $f(t)$ is the input to the following system

$$\ddot{x} + 2\dot{x} + 3x = f(t).$$

Show that the system is stable.

answer: We know that the general solution to the DE is

$$x(t) = x_p(t) + x_h(t).$$

Since we do not know what $f(t)$ we cannot find a particular solution x_p explicitly, but we can easily find the homogeneous solution

$$x_h(t) = c_1 \mathrm{e}^{-t}\cos(\sqrt{2}\,t) + c_2 \mathrm{e}^{-t}\sin(\sqrt{2}\,t).$$

Notice that exponents in $x_h(t)$ are negative, so no matter what the values of c_1 and c_2, the homogeneous solution $x_h(t)$ goes to 0 as t gets large. This tells us that as t gets large every solution $x(t)$ looks just like $x_p(t)$. That is, no matter what the initial conditions all solutions look alike in the long run. Since the long-term behavior of the solution does not depend on the initial conditions the system is by definition stable.

This example shows what can happen in general for a constant coefficient linear system $P(D)x = Q(D)f(t)$. We capture the result in a theorem:

Theorem. The system $P(D)x = Q(D)f(t)$ is stable exactly when all the characteristic roots have negative real part.

Since stability only depends on the homogeneous equation, a key point is the following

Stability is about the system not the input.

In fact, it only depends on $P(s)$ and not on $Q(s)$.

For stable systems we say that $x_h(t)$ is transient. The transient can be ignored in the long run, but in the short term it can be extremely important. For example, if the transient is so large it breaks the machine you may never reach the long term state.

Example 23.

1. $\dot{x} + 2x = f(t)$ is stable because $x_h = c\mathrm{e}^{-2t}$ goes to 0 as t gets large.

2. $P(D)x = \ddot{x} + 8\dot{x} + 7x = f(t)$ has characteristic roots -7, -1. Thus, the system is stable.

3. $P(D)x = \ddot{x} - 6\dot{x} + 25x = f(t)$ has characteristic roots $3 \pm 4i$. Since the real parts are positive the system is not stable.

4. A constant coefficient system with roots -2, -3, 4 is unstable because one of the roots has positive real part.

Since finding roots of polynomials can be difficult it is very useful to know that we can determine if a linear constant coefficient system is stable by criteria only involving its coefficients. For differential equations of order 1, 2, or 3 we will write down these criteria explicitly. For higher order equations we will simply reference the Routh-Hurwitz criteria.

Theorem. (Stability criteria for linear constant coefficient systems.) Consider the system

$$P(D)x = Q(D)f,$$

where we assume the the leading coefficient (coefficient of the highest order term) is always 1. If it is not 1, then we can first divide by it before applying the criteria below.

1. The system is stable if and only if the roots of $P(s)$ have negative real part.

2. The system is stable if and only if all solutions to the homogeneous equation $P(D)x = 0$ go asymptotically to 0.

3. If the system is first order then $P(s) = s + k$. So the root is $-k$. Thus the system is stable if and only if $k < 0$.

4. For a second order system $P(D)x = \ddot{x} + b\dot{x} + cx = Q(D)f$: stability is equivalent to $b > 0$ and $c > 0$. This is easy to prove by finding the roots explicitly

5. For a third order system with $P(s) = s^3 + as^2 + bs + c$, stability is equivalent to $a, b, c > 0$ and $ab > c$. This is proved using the Routh-Hurwitz criteria described below.

In general we will think of physical systems as having positive coefficients. So (3) says that all second order physical systems are stable. For third order systems, (4) says that being a physical system is not sufficient. Here is an example of an unstable third order system with all positive coefficients:

$$P(s) = (s+5)(s-1-100i)(s-1+100i) = s^3 + 3s^2 + 96s + 505.$$

You can see from the factored form that the roots are $1 \pm 100i$, -5, so the system is not stable. From the expanded form we see that $ab = 3 \cdot 96 < 505 = c$, so the Routh-Hurwitz criterion is not met.

6. For higher order systems there is the Routh-Hurwitz criteria, which you can read about at `http://math.mit.edu/~jorloff/suppnotes/suppnotes03/s.pdf` (There is also a link to a local copy of this note in the 'Review' section on the class website.)

6 Complex gain, gain and phase lag

In this section we need to consider the general LTI system

$$P(D)x = Q(D)f(t) \tag{7}$$

where $f(t)$ is considered the input.

There is no problem extending the Exponential Response Formula to the case. We give the derivation in the case $P(s) \neq 0$:

Theorem. (ERF) If $P(s) \neq 0$ then the equation

$$P(D)x = Q(D)\mathrm{e}^{st} \tag{8}$$

has a particular solution

$$x_p = \frac{Q(s)}{P(s)}\mathrm{e}^{st}.$$

Proof. Since $Q(D)\mathrm{e}^{st} = Q(s)\mathrm{e}^{st}$, Equation 8 can be written as

$$P(D)x = Q(s)\mathrm{e}^{st}.$$

Since $Q(s)$ is just a constant factor, linearity and the ERF tell us that $x_p = Q(s)\mathrm{e}^{st}/P(s)$ is a solution as claimed.

We will not need it, but an identical argument shows that $x_p = Q(s)t\mathrm{e}^{st}/P'(s)$ is a solution if $P(s) = 0$ and $P'(s) \neq 0$ etc.

It's also easy to extend the SRF to this system. For later use, we include a factor of F_0 in the input.

Theorem. (SRF) If $P(i\omega) \neq 0$ then the equation

$$P(D)x = Q(D)F_0\cos(\omega t) \tag{9}$$

has a particular solution

$$x_p = A\cos(\omega t - \phi), \text{ where } A = F_0|Q(i\omega)/P(i\omega)|, \text{ and } \phi = -\mathrm{Arg}(Q(i\omega)/P(i\omega)).$$

and the complex replacement equation

$$P(D)z_p = Q(D)F_0\mathrm{e}^{i\omega t} \tag{10}$$

has a particular solution

$$z_p = F_0 \frac{Q(i\omega)}{P(i\omega)} e^{i\omega t}.$$

We won't bother with a proof since it is essentially identical to the case $P(D)x = \cos(\omega t)$ done earlier.

In equation 9 we will consider $F_0 \cos(\omega t)$ to be the input. It will be useful to give names to all of the pieces in the SRF.

1. The amplitude of the input is F_0. This has the same units as the input quantity. It is the amplitude of the input sinusoid $F_0 \cos(\omega t)$.

2. The angular frequency of the input is ω. It has units of radians/time. Often we will be casual and refer to it as frequency, even though technically frequency should have units of cycles/time. Note that as ω changes the response changes.

3. The amplitude of the response is $A = F_0|Q(i\omega)/P(i\omega)|$. This has the same units as the response quantity. It is the amplitude of the sinusoidal response $A\cos(\omega t - \phi)$.

4. The gain is $g(\omega) = |Q(i\omega)/P(i\omega)|$. The gain is the factor that the input amplitude is multiplied by to get the amplitude of the response. It has the units needed to convert input units to output units.

5. The phase lag is $\phi = -\text{Arg}(Q(i\omega)/P(i\omega))$. The phase lag has units of radians, i.e. it's dimensionless. It is the part of one cycle that the output lags behind the input. For example, if $\phi = \pi/2$ then the peak of the output will be a quarter cycle behind that of the input.

6. The time lag is ϕ/ω. This has units of time. It is the time that peak of the output lags behind that of the input.

7. The complex gain is $Q(i\omega)/P(i\omega)$. This is the factor that the complex input is multiplied by to get the complex output.

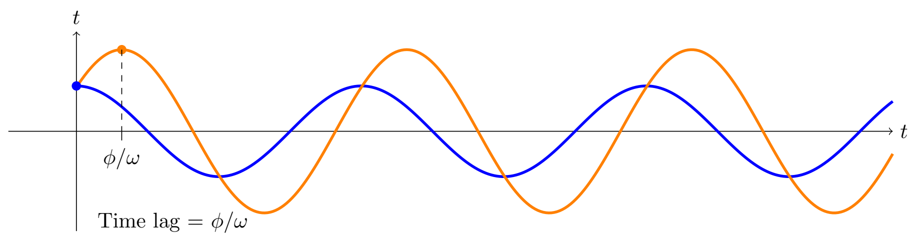
Important notes.

1. Everything named above is a function of ω.

2. The gain depends on what we designate as the input and output. For example, consider the equation

$$m\ddot{x} + b\dot{x} + kx = kF_0 \cos(\omega t).$$

This has periodic solution $x_p = \dfrac{kF_0}{|P(i\omega)|} \cos(\omega t - \phi)$. If we designate $F_0 \cos(\omega t)$ as the input and x as the output then the gain is $g(\omega) = \dfrac{k}{|P(i\omega)|}$. On the other hand if we designated $kF_0 \cos(\omega t)$ to be the input then the gain is $g(\omega) = \dfrac{1}{|P(i\omega)|}$.



Smaller (blue) sinusoid = input; larger, shifted (orange) sinusoid = output