Fourier Cosine Series Examples

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It is an remarkable fact that (almost) any function can be expressed as an infinite sum of cosines, the *Fourier cosine series*. For a function $f(x)$ defined on $x \in [0, \pi]$, one can write $f(x)$ as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

for some coefficients a_k. We can compute the a_ℓ very simply: for any given ℓ, we integrate both sides against $\cos(\ell x)$. This works because of orthogonality: $\int_0^\pi \cos(kx)\cos(\ell x)dx$ can easily be shown to be zero unless $k = \ell$ (just do the integral). Plugging the above sum into $\int_0^\pi f(x)\cos(\ell x)dx$ therefore gives zero for $k \neq \ell$ and $\int_0^\pi \cos^2(\ell x) = \pi/2$ for $k = \ell$, resulting in the equation

$$a_\ell = \frac{2}{\pi}\int_0^\pi f(x)\cos(\ell x)\,dx.$$

Fourier claimed (without proof) in 1822 that *any* function $f(x)$ can be expanded in terms of cosines in this way, even discontinuous functions. This turned out to be false for various badly behaved $f(x)$, and controversy over the exact conditions for convergence of the Fourier series lasted for well over a century, until the question was finally settled by Carleson (1966) and Hunt (1968): any function $f(x)$ where $\int |f(x)|^{1+\varepsilon}dx$ is finite for some $\varepsilon > 0$ has a Fourier series that converges *almost everywhere* to $f(x)$ [except possibly at isolated points of discontinuities]. At points where $f(x)$ has a jump discontinuity, the Fourier series converges to the midpoint of the jump. So, as long as one does not care about crazy divergent functions or the function value exactly at points of discontinuity (which usually has no practical significance), Fourier's remarkable claim is essentially true.

Example

To illustrate the convergence of the cosine series, let's consider an example. Let's try $f(x) = x$, which seems impossible to expand in cosines because cosines all have zero slope at $x = 0$ whereas $f'(0) = 1$. Nevertheless, it has a convergent cosine series that can be computed via integration by parts:

$$a_k = \frac{2}{\pi}\int_0^\pi x\cos(kx)dx = \frac{2}{\pi k}x\sin(kx)\Big|_0^\pi - \frac{2}{\pi k}\int_0^\pi \sin(kx)dx = \begin{cases} 0 & k \text{ even} \\ -\frac{4}{\pi k^2} & k \text{ odd} \end{cases}.$$

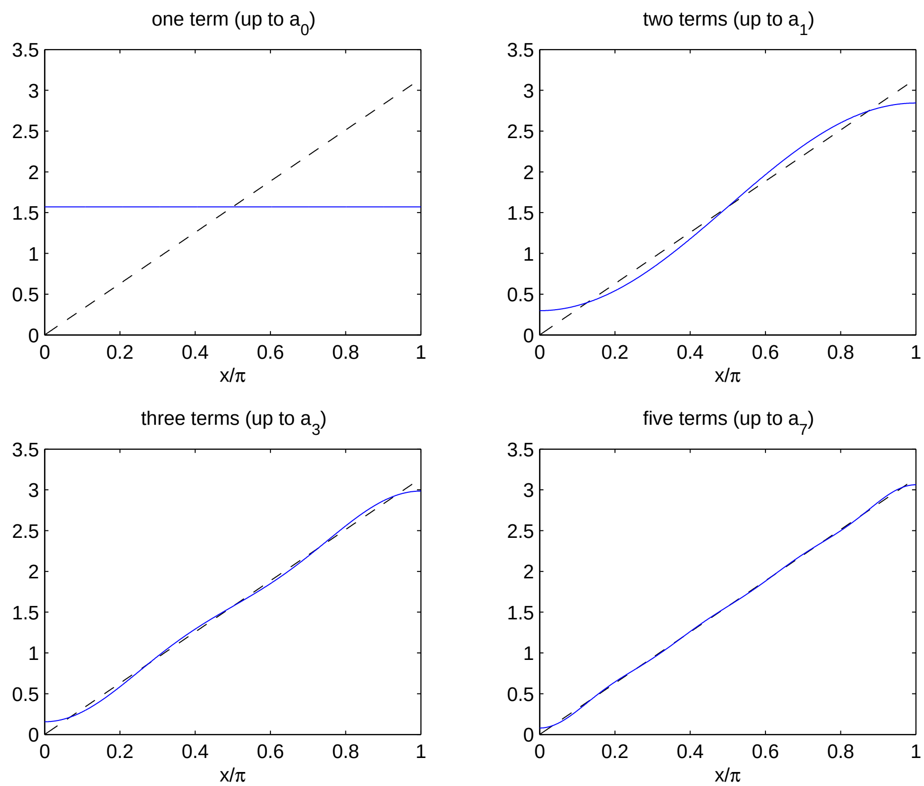

Figure 1: Fourier cosine series (blue lines) for the function $f(x) = x$ (dashed black lines), truncated to a finite number of terms (from 1 to 5), showing that the series indeed converges everywhere to $f(x)$.

We divided by 0 for $k = 0$ in the above integral, however, so we have to compute a_0 separately: $a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \pi$. The resulting cosine-series expansion is plotted in figure 1, truncated to 1, 2, 3, or 5 terms in the series. Already by just five terms, you can see that the cosine series is getting quite close to $f(x) = x$. Mathematically, we can see that the series coefficients a_k decrease as $1/k^2$ asymptotically, so higher-frequency terms have smaller and smaller contributions.

General convergence rate

Actually, this $1/k^2$ decline of a_k is typical for any function $f(x)$ that does not have zero slope at $x = 0$ and $x = \pi$ like the cosine functions, as can be seen via integration by parts:

$$
\begin{aligned}
a_k = \frac{2}{\pi}\int_0^\pi f(x)\cos(kx)dx &= \frac{2}{\pi k} f(x)\sin(kx)\Big|_0^\pi - \frac{2}{\pi k}\int_0^\pi f'(x)\sin(kx)dx \\
&= \frac{2}{\pi k^2} f'(x)\cos(kx)\Big|_0^\pi - \frac{2}{\pi k^2}\int_0^\pi f''(x)\cos(kx)dx \\
&= -\frac{2}{\pi k^2}[f'(0) \pm f'(\pi)] - \frac{2}{\pi k^3} f''(x)\sin(kx)\Big|_0^\pi + \frac{2}{\pi k^3}\int_0^\pi f'''(x)\sin(kx)dx \\
&= -\frac{2}{\pi k^2}[f'(0) \pm f'(\pi)] + \frac{2}{\pi k^4}[f'''(0) \pm f'''(\pi)] - \cdots,
\end{aligned}
$$

where the $\pm$ is $-$ for even k and $+$ for odd k. Thus, we can see that, unless $f(x)$ has zero first derivative at the boundaries, a_k decreases as $1/k^2$ asymptotically. If $f(x)$ has zero first derivative, then a_k decreases as $1/k^4$ unless $f(x)$ has zero *third* derivative at the boundaries (like cosine). If $f(x)$ has zero first and third derivatives, then a_k decreases like $1/k^6$ unless $f(x)$ has zero fifth derivative, and so on. (Of course, in all of the above we assumed that $f(x)$ was infinitely differentiable in the interior of the integration region.) If *all* of the odd derivatives of $f(x)$ are zero at the endpoints, then a_k decreases asymptotically faster than *any* polynomial in $1/k$ — typically in this case, a_k decreases exponentially fast.[1]

[1] Technically, to get a_k decreasing exponentially fast (or occasionally faster), we need $f(x)$ to have zero odd derivatives at the endpoints and be an "analytic" function (i.e., having a convergent Taylor series) in a neighborhood of $[0, \pi]$ in the complex x plane . Analyzing this properly requires complex analysis (18.04), however.