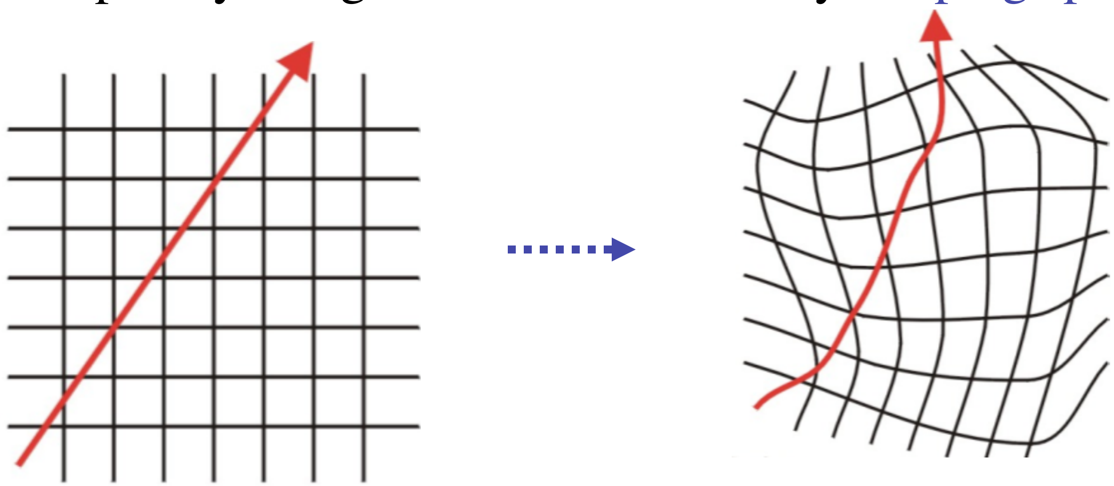
A beautiful approach:
"Transformational optics"

[several precursors, but generalized & popularized by Ward & Pendry (1996)]

warp a ray of light …by warping space(?)

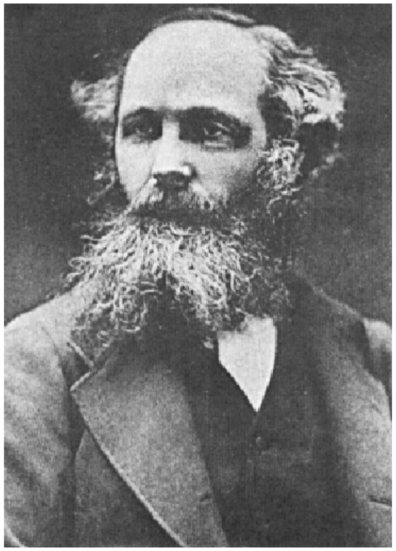
[figure: J. Pendry]



Euclidean **x** coordinates transformed **x**'(**x**) coordinates

amazing fact:

Solutions of ordinary Euclidean Maxwell equations in **x**'
= transformed solutions from **x**
if **x**' uses *transformed materials* ε' and μ'



James Clerk Maxwell
1864

Maxwell's Equations

constants: ε_0, μ_0 = vacuum permittivity/permeability = 1
c = vacuum speed of light = $(\varepsilon_0 \mu_0)^{-1/2}$ = 1

Gauss:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

Ampere:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Faraday:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

constitutive relations:

$$\mathbf{E} = \mathbf{D} - \mathbf{P}$$

$$\mathbf{H} = \mathbf{B} - \mathbf{M}$$

electromagnetic fields:

E = electric field
D = displacement field
H = magnetic field / induction
B = magnetic field / flux density

sources: **J** = current density
ρ = charge density

material response to fields:
P = polarization density
M = magnetization density

Constitutive relations for macroscopic linear materials

$\mathbf{P} = \chi_e \mathbf{E}$ $\Rightarrow$ $\mathbf{D} = (1+\chi_e)\,\mathbf{E} = \varepsilon\,\mathbf{E}$

$\mathbf{M} = \chi_m \mathbf{H}$ $\quad$ $\mathbf{B} = (1+\chi_m)\,\mathbf{H} = \mu\,\mathbf{H}$

where $\varepsilon = 1+\chi_e$ = electric permittivity
or dielectric constant
$\mu = 1+\chi_m$ = magnetic permeabili

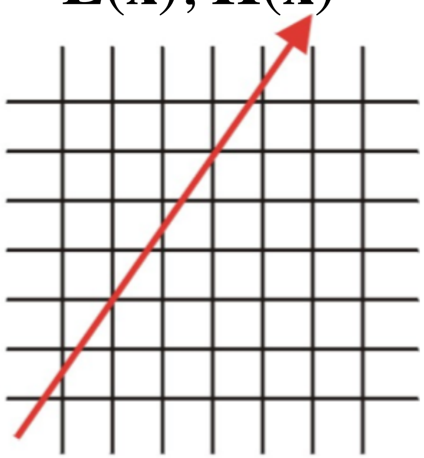
$\varepsilon\mu = (\text{refractive index})^2$

Transformation-mimicking materials

[Ward & Pendry (1996)]

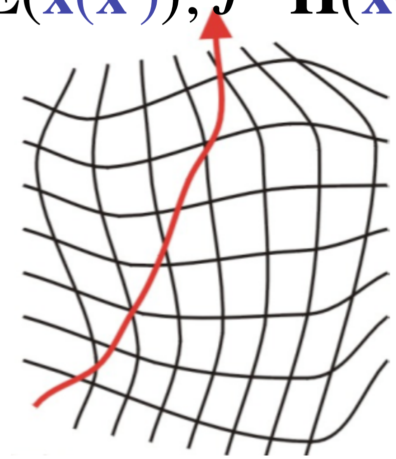
[figure:
J. Pendry]

$\mathbf{E}(\mathbf{x}), \mathbf{H}(\mathbf{x})$



Euclidean **x** coordinates

$J^{-T}\mathbf{E}(\mathbf{x}(\mathbf{x}')), J^{-T}\mathbf{H}(\mathbf{x}(\mathbf{x}'))$



transformed **x'**(**x**) coordinates

$\varepsilon(\mathbf{x}), \mu(\mathbf{x})$

(*linear* materials)

$$\varepsilon' = \frac{J\varepsilon J^T}{\det J} \quad , \quad \mu' = \frac{J\mu J^T}{\det J}$$

J = Jacobian ($J_{ij} = \partial x_i' / \partial x_j$)

(isotropic, nonmagnetic [μ=1], homogeneous materials

⇒ anisotropic, magnetic, inhomogeneous materials)

an elementary derivation

[Kottke (2008)]

$$\partial'_i H'_j \epsilon_{ijk} = \frac{1}{\det \boldsymbol{\mathcal{J}}} \mathcal{J}_{kc} \varepsilon_{cd} \mathcal{J}_{ld} \frac{\partial E'_l}{\partial t} + \frac{\mathcal{J}_{kc} J_c}{\det \boldsymbol{\mathcal{J}}}$$

$$\nabla' \times \mathbf{H}' = \frac{\boldsymbol{\mathcal{J}} \boldsymbol{\varepsilon} \boldsymbol{\mathcal{J}}^T}{\det \boldsymbol{\mathcal{J}}} \frac{\partial \mathbf{E}'}{\partial t} + \mathbf{J}'$$

ε'

$$\mathcal{J}_{ij} = \frac{\partial x'_i}{\partial x_j}$$

Jacobian

$$\partial_a = \mathcal{J}_{ba} \partial'_b$$

chain rule

$$\begin{aligned} E_a &= \mathcal{J}_{ba} E'_b, \\ H_a &= \mathcal{J}_{ba} H'_b. \end{aligned}$$

choice of fields
$\mathbf{E}'$, $\mathbf{H}'$ in $\mathbf{x}'$

Cloaking transformations

[Pendry, Schurig, & Smith, *Science* **312**, 1780 (2006)]

virtual space

d

ambient ε_a, μ_a

$\mathbf{x}' \neq \mathbf{x}$

$\mathbf{x}' = \mathbf{x}$

ambient ε_a, μ_a

real space

diameter D

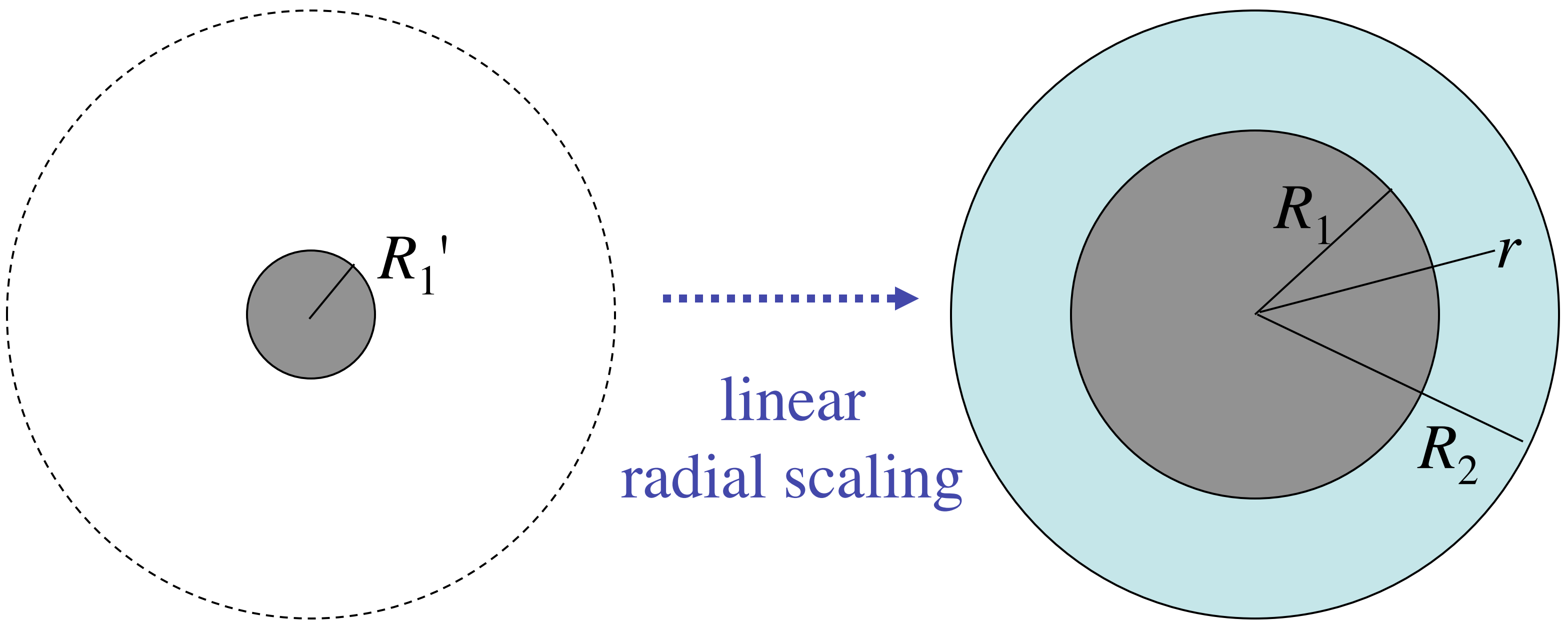
cloaking materials ε', μ'

ambient ε_a, μ_a

perfect cloaking: $d \rightarrow 0$

(= *singular* transformation J)

Example: linear, spherical transform



cloak materials: $$\varepsilon_\theta/\varepsilon_a = \mu_\theta/\mu_a = \varepsilon_\phi/\varepsilon_a = \mu_\phi/\mu_a = \frac{R_2 - R_1'}{R_2 - R_1}$$

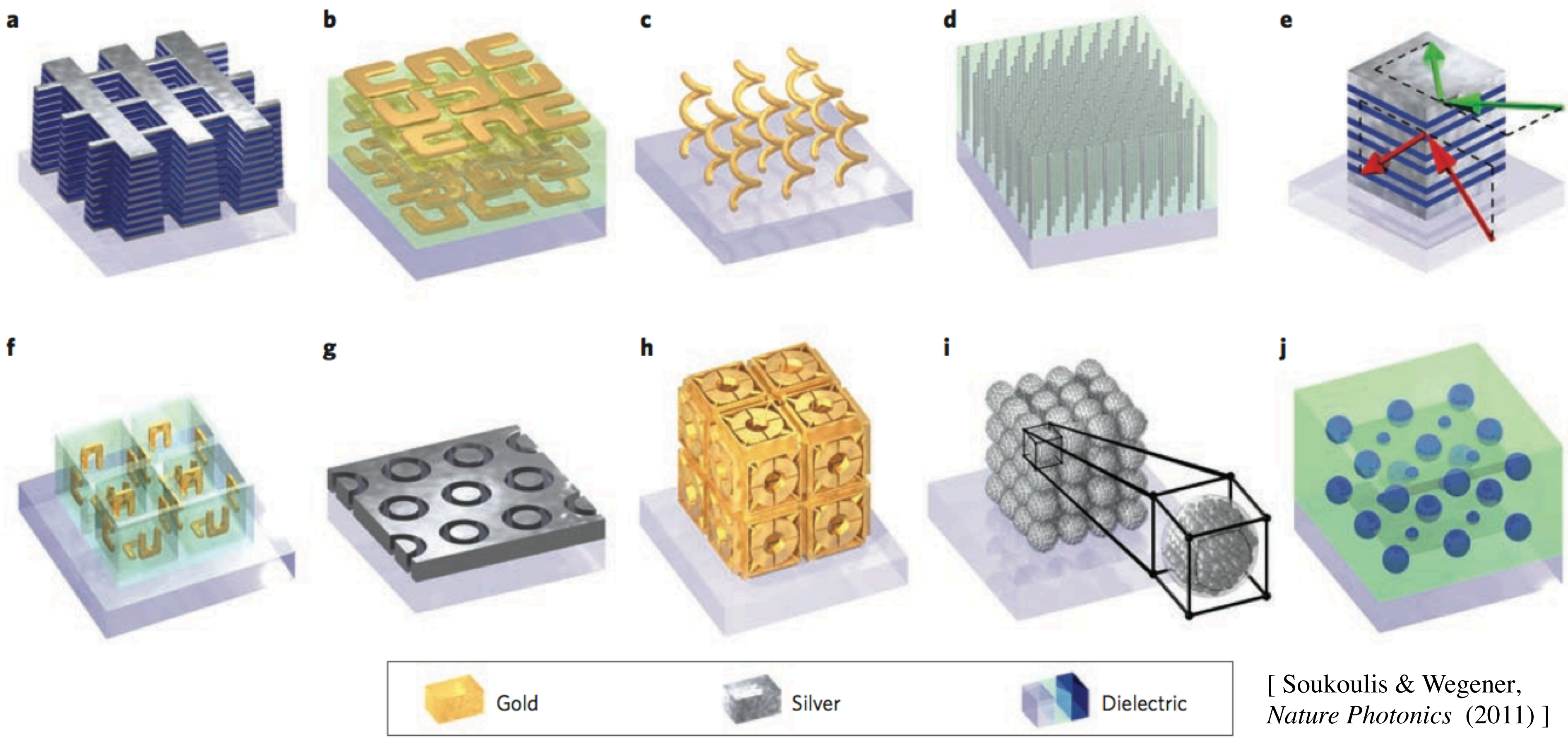
$$\varepsilon_r/\varepsilon_a = \mu_r/\mu_a = \frac{R_2 - R_1}{R_2 - R_1'} \frac{\left[R_1' + \frac{r - R_1}{R_2 - R_1}(R_2 - R_1')\right]^2}{r^2}$$

$= 0$
at $r = R_1$
for $R_1' = 0$

[note: no "negative index" $\varepsilon, \mu < 0$]

Are these materials attainable?

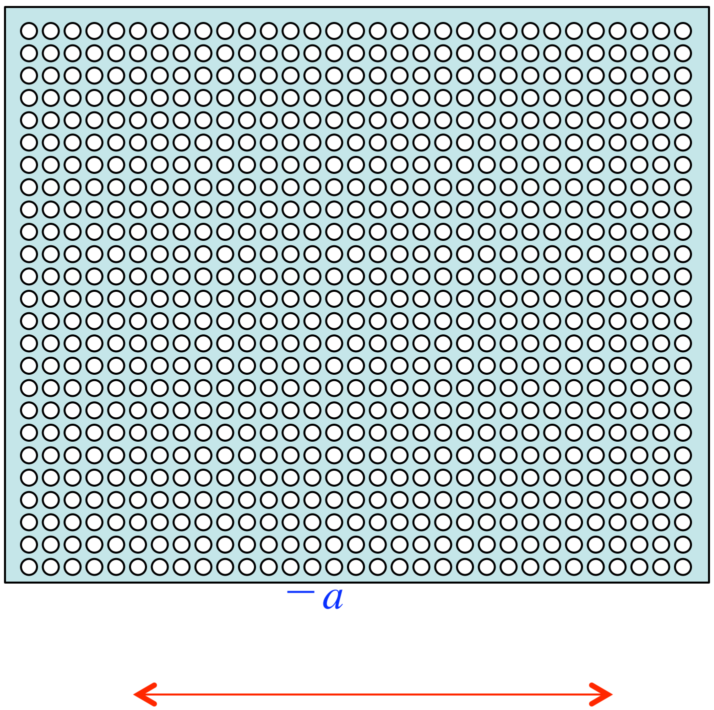
Highly anisotropic, even (effectively) magnetic materials can be fabricated by a "metamaterials" approach:



[Soukoulis & Wegener, *Nature Photonics* (2011)]

$\lambda >>$ microstructure $\Rightarrow$ "effective" *homogeneous* ε, μ = "metamaterial"

Simplest Metamaterial: "Average" of two dielectrics



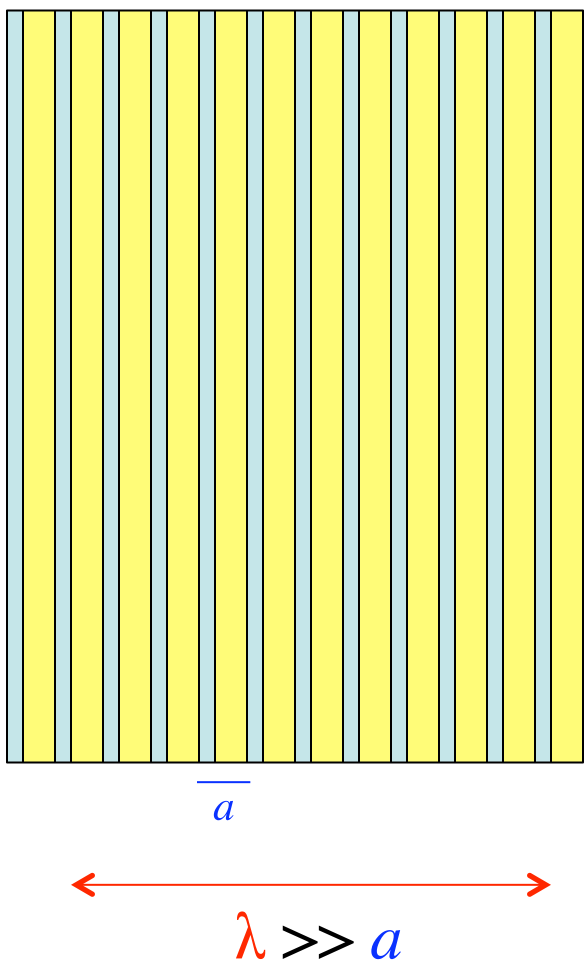
$\lambda >> a$

effective dielectric is just some average, subject to Weiner bounds (Aspnes, 1982) in the large-λ limit:

$$\left\langle \varepsilon^{-1} \right\rangle^{-1} \leq \varepsilon_{\text{effective}} \leq \left\langle \varepsilon \right\rangle$$

(isotropic for sufficient symmetry)

Simplest anisotropic metamaterial: multilayer film in large-λ limit



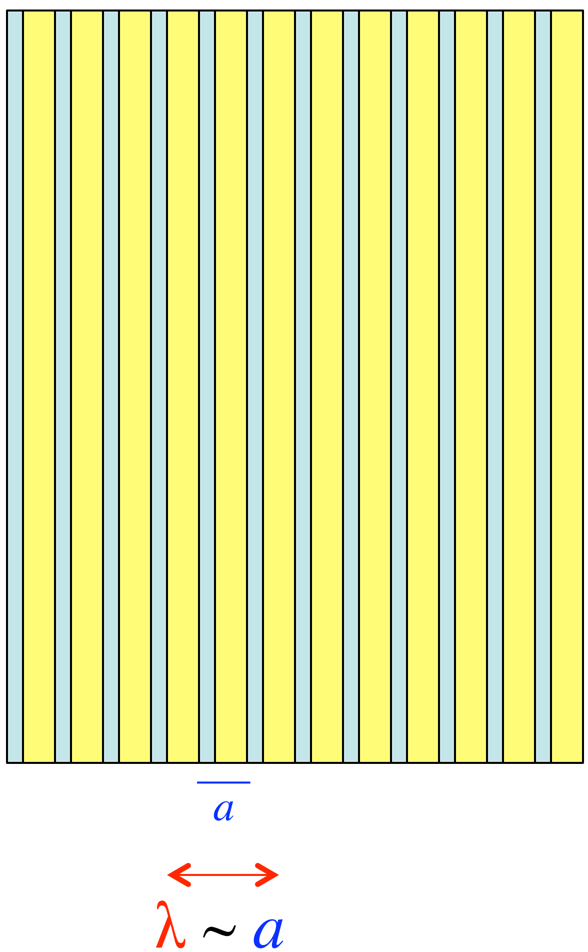
$$\varepsilon_{ij}^{\text{eff}} = \frac{\langle D_i \rangle}{\langle E_j \rangle} = \frac{\langle \varepsilon E_i \rangle}{\langle E_j \rangle} = \frac{\langle D_i \rangle}{\langle \varepsilon^{-1} D_j \rangle}$$

key to anisotropy is differing continuity conditions on **E**:

↑ $E_{\|}$ continuous ⇒ $\varepsilon_{\|} = \langle\varepsilon\rangle$

→ $D_{\perp} = \varepsilon E_{\perp}$ continuous ⇒ $\varepsilon_{\perp} = \langle\varepsilon^{-1}\rangle^{-1}$

[Not a metamaterial: multilayer film in $\lambda \sim a$ regime]



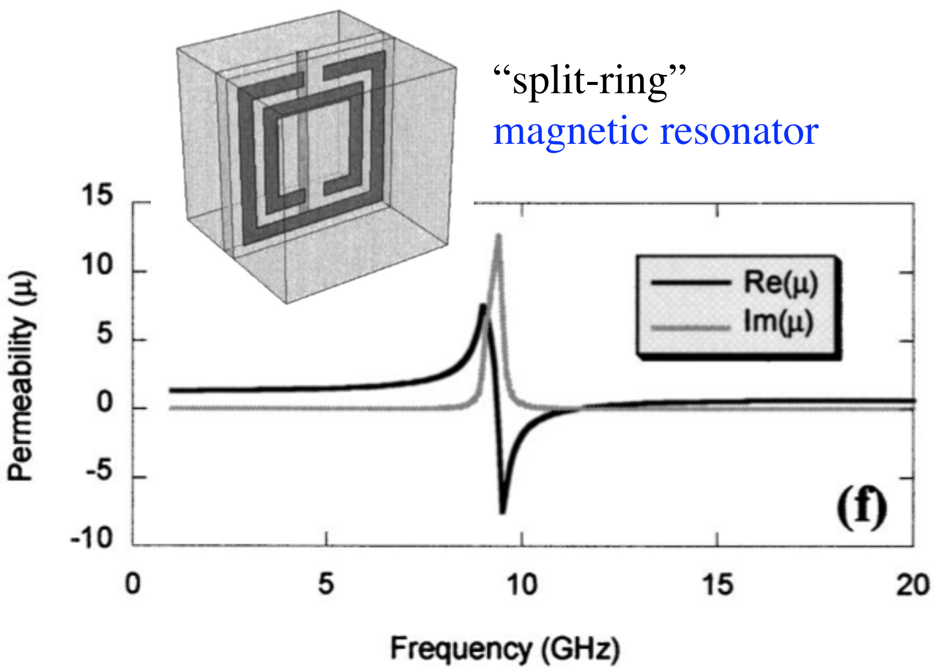
e.g. Bragg reflection regime
(photonic bandgaps) for $\lambda \sim 2a$
is not completely reproduced
by any effective ε, μ

Metamaterials are a special case
of periodic electromagnetic media
(photonic crystals)

“Exotic” metamaterials

[= properties *very different* from constituents]

from sub-λ metallic resonances



[Smith et al, *PRE* (2005)]

resonance
= pole in polarizability χ

$$\chi \sim \frac{\#}{\omega - (\omega_0 - i\Gamma_0)}$$

($\Gamma_0 > 0$ for causal, passive)

Problem: more exotic often
= more absorption

Problem: metals quite lossy
@ optical & infrared