18.369 Problem Set 3

Due Friday, 11 March 2016.

Problem 1: Periodic waveguides

In class, we showed by a variational proof that any $\varepsilon(y)$, in two dimensions, gives rise to at least one guided mode whenever $\varepsilon(y)^{-1} = \varepsilon_{\text{lo}}^{-1} - \Delta(y)$ for $\int \Delta > 0$ and $\int |\Delta| < \infty$.[1] At least, we showed it for the TE polarization ($\mathbf{H}$ in the $\hat{\mathbf{z}}$ direction). Now, you will show the same thing much more generally, but using the same basic technique.

(a) Let $\varepsilon(x,y)^{-1} = 1 - \Delta(x,y)$ be a periodic function $\Delta(x,y) = \Delta(x+a,y)$, with $\int |\Delta| < \infty$ and $\int_0^a \int_{-\infty}^{\infty} \Delta(x,y)dxdy > 0$. Prove that at least one TE guided mode exists, by choosing an appropriate (simple!) trial function of the form $\mathbf{H}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$. That is, show by the variational theorem that $\omega^2 < c^2k^2$ for the lowest-frequency eigenmode. (It is sufficient to show it for $|k| \le \pi/a$, by periodicity in k-space; for $|k| > \pi/a$, the light line is not $\omega = c|k|$.)

(b) Prove the same thing as in (a), but for the TM polarization ($\mathbf{E}$ in the $\hat{\mathbf{z}}$ direction). Hint: you will need to pick a trial function of the form $\mathbf{H}(x,y) = [u(x,y)\hat{\mathbf{x}} + v(x,y)\hat{\mathbf{y}}]e^{ikx}$ where u and v are some (simple!) functions such that $\nabla \cdot \mathbf{H} = 0$.[2]

Problem 2: Point sources & periodicity

Suppose we are in 2d (xy plane), working with the TM polarization ($\mathbf{E}$ out of plane), and have a periodic (period a) surface shown in Fig 1(left). Above the surface is a time-harmonic point source $\mathbf{J} = \delta(x)\delta(y)e^{-i\omega t}\hat{z}$ (choosing the origin to be the location of the point source, for convenience). As you saw in pset 2, you can define a frequency-domain problem $(\nabla \times \nabla \times -\omega^2\varepsilon)\mathbf{E} = i\omega\mathbf{J}$ (setting $\mu_0 = \varepsilon_0 = 1$ for convenience) for the time-harmonic fields in response to this current.

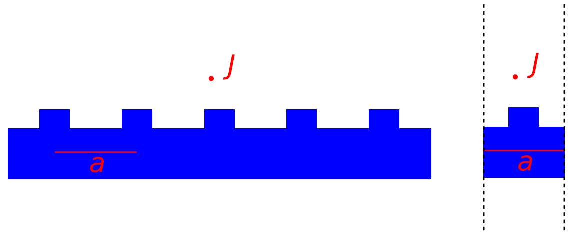

Figure 1: Schematic for problem 1. *Left*: a time-harmonic point source $\mathbf{J}$ above a periodic surface. *Right:* the problem can be reduced to solving a set of problems with point sources in a single unit cell, with periodic boundary conditions on the fields.

In this problem, you will explain how to take advantage of the fact that the structure (but not the source or fields!) is periodic, by reducing it to a set of problems of the form shown in Fig. 1(right): solving for the fields of the *same* point source $\mathbf{J}$, but in a *single unit cell* of the structure with *Bloch-periodic boundary conditions* on the fields.

(a) Show that the total resulting electric field $\mathbf{E}$can be written as a superposition of solutions $\mathbf{E}_k$ to $(\nabla \times \nabla \times -\omega^2\varepsilon)\mathbf{E}_k = i\omega\mathbf{J}$ in a unit-cell domain with Bloch-periodic boundary conditions. Hint:

$$\delta(x) = \frac{a}{2\pi}\int_0^{2\pi/a}\left[\sum_{n=-\infty}^{\infty}\delta(x-na)e^{ikna}\right]dk$$

and recall conservation of irrep.

(b) Suppose that we want to compute the radiated power P (per unit z) from $\mathbf{J}$ by integrating the Poynting flux through a plane above the current ($y = y_0 > 0$):

$$P = \frac{1}{2}\int_{-\infty}^{\infty}\hat{y}\cdot\Re\left[\mathbf{E}^*(x,y_0)\times\mathbf{H}(x,y_0)\right]dx.$$

Show that $P = \frac{a}{2\pi}\int_0^{2\pi/a}P_k dk$, a simple integral of powers P_k computed *separately* for each periodic subproblem above. (Hint: orthogonality of partner functions.)

[1] As in class, the latter condition on Δ will allow you to swap limits and integrals for any integrand whose magnitude is bounded above by some constant times $|\Delta|$ (by Lebesgue's dominated convergence theorem).

[2] You might be tempted, for the TM polarization, to use the $\mathbf{E}$ form of the variational theorem that you derived in problem 1, since the proof in that case will be somewhat simpler: you can just choose $\mathbf{E}(x,y) = u(x,y)e^{ikx}\hat{\mathbf{z}}$ and you will have $\nabla \cdot \varepsilon\mathbf{E} = 0$ automatically. However, this will lead to an inequivalent condition $\int(\varepsilon - 1) > 0$ instead of $\int \Delta = \int \frac{\varepsilon-1}{\varepsilon} > 0$.

Problem 3: Waveguides in MPB

For this problem, you will gain some initial experience with the MPB numerical eigensolver described in class, and which is available on Athena. Refer to the class handouts, and also to the online MPB documentation at jdj.mit.edu/mpb/doc. For this problem, you will study the simple 2d dielectric waveguide (with $\varepsilon_{hi} = 12$) that you analyzed analytically above, along with some variations thereof—start with the sample MPB input file (2dwaveguide.ctl) that was introduced in class and is available on the course web page.

(a) Plot the TM (E_z) even modes as a function of k, from $k = 0$ to a large enough k that you get at least four modes. Compare your numerical calculation to the analytical prediction, quoted below, for the "cutoff" k values where new modes should appear. Show what happens to this "cutoff point" when you change the size of the computational cell.

Analytically, one can show that you should get a new even mode for a waveguide of width h and contrast $f = \varepsilon_{lo}/\varepsilon_{hi} < 1$ when $kh/2\pi$ an integer multiple of $1/\sqrt{1/f - 1}$. Here, $h = a = 1$, and $f = 1/12$, so should get modes starting at $ka/2\pi$ of approximately 0.3015, 0.6030, and 0.9045.

(b) Plot the fields of some guided modes on a log scale, and verify that they are indeed exponentially decaying away from the waveguide. (What happens at the computational cell boundary?)

(c) Modify the structure so that the waveguide has $\varepsilon = 2.25$ instead of air on the $y < -h/2$ side. Show that there is a low-ω cutoff for the TM guided bands, and find the cutoff frequency. (There is a general argument that an asymmetric waveguide "cladding" of this sort leads to low-frequency cutoffs.)

(d) Create the waveguide with the following profile:

$$\varepsilon(y) = \begin{cases} 2 & 0 \le y < h/2 \\ 0.8 & -h/2 < y < 0 \\ 1 & |y| \ge h/2 \end{cases} .$$

Should this waveguide have a guided mode as $k \to 0$? Show numerical evidence to support your conclusion (careful: as the mode becomes less localized you will need to increase the computational cell size).

Problem 4: Band gaps in MPB

Consider the 1d periodic structure consisting of two alternating layers: $\varepsilon_1 = 12$ and $\varepsilon_2 = 1$, with thicknesses d_1 and $d_2 = a - d_1$, respectively. To help you with this, I've created a sample input file *bandgap1d.ctl* that is posted on the course web page.

(a) Using MPB, compute and plot the fractional TM gap size (of the *first* gap, i.e lowest ω) vs. d_1 for d_1 ranging from 0 to a. What d_1 gives the largest gap? Compare to the "quarter-wave" thicknesses $d_{1,2} = a\sqrt{\varepsilon_{2,1}}/[\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}]$ (see section "size of the band gap" in chapter 4 of the book).

(b) Given the optimal parameters above, what would be the physical thicknesses in order for the mid-gap vacuum wavelength to be $\lambda = 2\pi c/\omega = 1.55\mu$m? (This is the wavelength used for most optical telecommunications.)

(c) Plot the 1d TM band diagram for this structure, with d_1 given by the quarter wave thickness, showing the first five gaps. Also compute it for $d_1 = 0.12345$ (which I just chose randomly), and superimpose the two plots (plot the quarter-wave bands as solid lines and the other bands as dashed). What special features does the quarter-wave band diagram have?