18.369 Problem Set 2

Due Friday, 26 February 2016.

Problem 1: Projection operators

(You the Great Orthogonality Theorem and/or the various orthogonality results for the rows/columns of the character table.)

(a) The representation-theory handout gives a formula for the projection operator from a state onto its component that transforms as a particular representation. Prove the correctness of this formula: in particular, for any function ψ, show that $\hat{P}_i^{(\alpha)}\psi$ transforms as the i-th partner function representation α. (You can use the fact, from class, that we showed that any function can be decomposed into a sum of partner functions of irreducible representations. Consider what $\hat{P}_i^{(\alpha)}$ does to one of thes partner functions.)

(b) Prove that the sum of the projection operators $\hat{P}^{(\alpha)}$ over all the irreducible representations α gives $\sum_\alpha \hat{P}^{(\alpha)} = \sum_{\alpha,i} \hat{P}_i^{(\alpha)} = 1$ (the identity operator), by using the column-orthogonality property of the character table.

(c) Prove that the projection operators $\hat{P}_i^{(\alpha)}$ are Hermitian for any unitary representation $D^{(\alpha)}$, assuming that $\hat{O}_g$ is unitary for all $g \in G$ (unitarity of $\hat{O}_g$ is trivial to prove for symmetry groups where g is a rotation and/or translation).

(d) Using the previous parts, prove that $\langle \phi_i^{(\alpha)}, \psi_j^{(\beta)} \rangle = 0$ if $\phi_i^{(\alpha)}$ and $\psi_j^{(\beta)}$ are partner functions of different irreducible representations $\alpha \neq \beta$, or if they correspond to different components $i \neq j$ of the same representation $\alpha = \beta$. (Hint: insert *1* into the inner product.) (Assume unitary irreps.)

Problem 2: A square metal box

In class, we considered a two-dimensional (xy) problem of light in an $L \times L$ square of air ($\varepsilon = 1$) surrounded by perfectly conducting walls (in which $\mathbf{E} = 0$). We solved the case of $\mathbf{H} = H_z(x,y)\hat{\mathbf{z}}$ and saw solutions corresponding to five different representations of the symmetry group ($C_{4\mathrm{v}}$).

(a) Solve for the eigenfunctions of the other polarization: $\mathbf{E} = E_z(x,y)\hat{\mathbf{z}}$ (you will need the $\mathbf{E}$ eigenproblem from problem set 1), with the boundary condition that $E_z = 0$ at the metal walls.

(b) Sketch (or plot on a computer) and classify these solutions according to the representations of $C_{4\mathrm{v}}$ enumerated in class. (Like in class, you will get some reducible accidental degeneracies.) Look at enough solutions to find *all five* irreps, and to illustrate the general pattern (you should find that the irreps appear in repeating patterns).

Problem 3: A triangular metal box

Consider the two-dimensional solutions in a *triangular* perfect-metal box with side L. Don't try to solve this analytically; instead, you will use symmetry to sketch out what the possible solutions will look like for both E_z and H_z polarizations.

(a) List the symmetry operations in the space group (choose the origin at the center of the triangle so that the space group is symmorphic), and break them into conjugacy classes. (This group is traditionally called $C_{3\mathrm{v}}$). Verify that the group is closed under composition (i.e. that the composition of two operations always gives another operation in the group) by giving the "multiplication table" of the group (whose rows and columns are group members and whose entries give their composition).

(b) Find the character table of $C_{3\mathrm{v}}$, using the rules from the representation-theory handout.

(c) Give unitary representation matrices D for each irreducible representation of $C_{3\mathrm{v}}$.

(d) Sketch possible $\omega \neq 0$ E_z and H_z solutions that would transform as these representations. What representation should the lowest-ω mode (excluding $\omega = 0$) of each polarization correspond to?

(e) If there are any (non-accidental) degenerate modes, show how given one of the modes we can get the other orthogonal eigenfunction(s) (e.g. in the square case we could get one from the 90° rotation of the other for a degenerate pair, but the triangular structure is not symmetric under 90° rotations). Hint: use your representation matrices.

Problem 4: Cylindrical symmetry

Suppose that we have a *cylindrical* metallic waveguide—that is, a perfect metallic tube with radius R, which is uniform in the z direction. The interior of the tube is simply air ($\varepsilon = 1$).

(a) This structure has continuous rotational symmetry around the z axis, which is called the C_∞ group.[1] Find the irreducible representations of this group (there are infinitely many because it is an infinite group).

(b) For simplicity, consider the (Hermitian) *scalar* wave equation $-\nabla^2\psi = \frac{\omega^2}{c^2}\psi$ with $\psi|_{r=R} = 0$. Show that, when we look for solutions ψ that transform like one of the representations of the C_∞ group from above, and have z dependence e^{ikz} (from the translational symmetry), then we obtain a Bessel equation (Google it if you've forgotten Mr. Bessel). Write the solutions in terms of Bessel functions, assuming that you are given their zeros $x_{m,n}$ (i.e. $J_m(x_{m,n}) = 0$ for $n = 1, 2, \ldots$, where J_m is the Bessel function of the first kind...if you Google for "Bessel function zeros" you can find them tabulated). Sketch the dispersion relation $\omega(k)$ for a few bands.

(c) From the general orthogonality of Hermitian-operator eigenfunctions, derive/prove an orthogonality integral for the Bessel functions. (No, just looking one up on Wikipedia doesn't count.)

[1] It also has an infinite set of mirror planes containing the z axis, but let's ignore these for now. If they are included, the group is called $C_{\infty\text{v}}$.

Problem 5: Conservation laws

Suppose that we introduce a nonzero current $\mathbf{J}(\mathbf{x})e^{-i\omega t}$ into Maxwell's equations at a given frequency ω, and we want to find the resulting time-harmonic electric field $\mathbf{E}(\mathbf{x})e^{-i\omega t}$.

(a) Show that this results in a linear equation of the form $\hat{A}\mathbf{E} = \mathbf{b}(\mathbf{x})$, where $\hat{A}$ is some linear operator and $\mathbf{b}$ is some known right-hand side proportional to the current density $\mathbf{J}$.

(b) Prove that, if $\mathbf{J}$ transforms as some irreducible representation of the space group then $\mathbf{E}$ (which you can assume for now is a *unique* solution) does also. (This is the analogue of the conservation in *time* that we showed in class, except that now we are proving it in the frequency domain. You could prove it by Fourier-transforming the theorem from class, I suppose, but do *not* do so—instead, prove it directly from the linear equation here.)

(c) Formally, $\mathbf{E} = \hat{A}^{-1}\mathbf{b}$, where $\hat{A}^{-1}$ is related to the *Green's function* of the system. What happens if ω is one of the eigenfrequencies? (No rigorous solution required, just a few words about what you expect to happen physically in such a case; you can suppose for simplicity that you have a problem in a finite domain with impenetrable walls, e.g. PEC boundary conditions.)