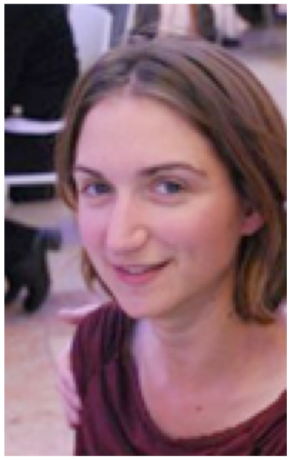
The Mathematics of Lasers

Steven G. Johnson
MIT Applied Mathematics



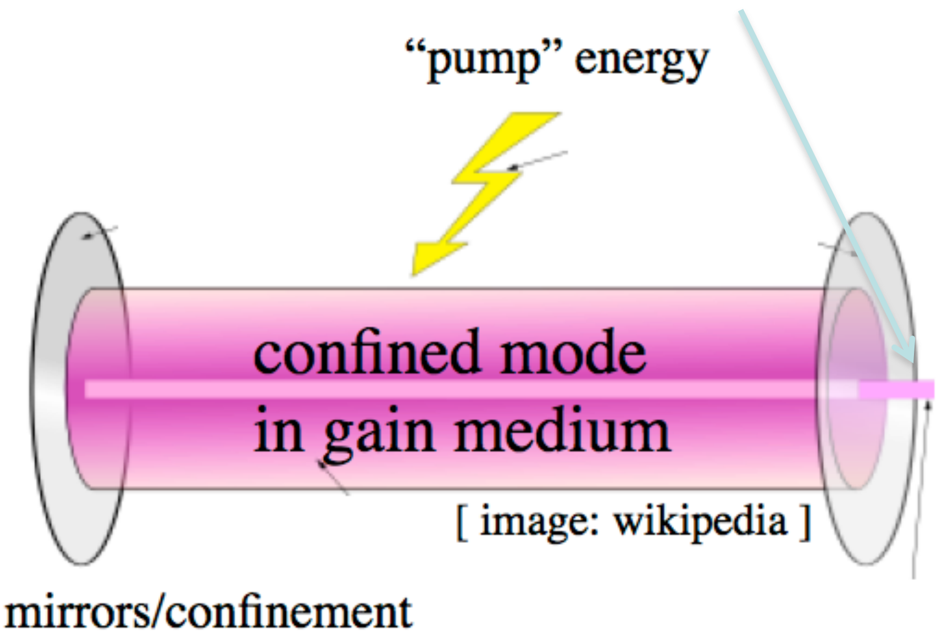
Adi Pick (Harvard), David Liu (MIT),

Sofi Esterhazy, M. Liertzer, K. Makris, M. Melenck, S. Rotter (Vienna),
Alexander Cerjan & A. Doug Stone (Yale),
Li Ge (CUNY), Yidong Chong (NTU)

What is a laser?

- a laser is a resonant cavity…
- with a gain medium…
- pumped by external power source

population inversion → stimulated emission



1d laser:
light bouncing
between 2 mirrors

Resonance

an oscillating mode trapped for a long time in some volume

(of light, sound, …)

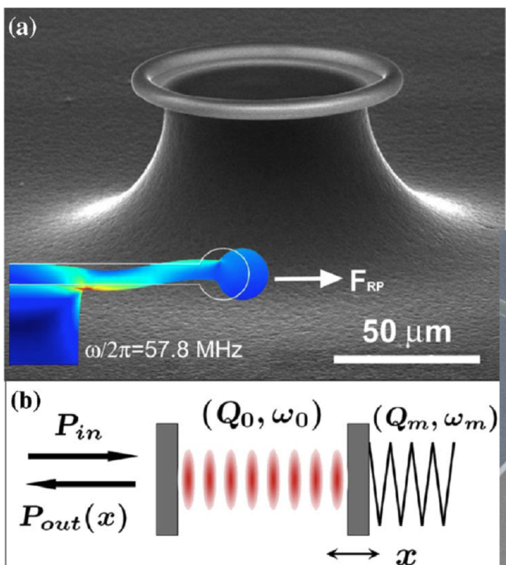
lifetime $\tau \gg 2\pi/\omega_0$

frequency ω_0

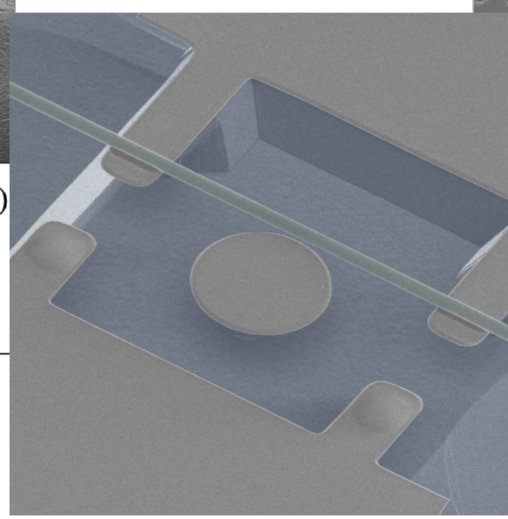
quality factor $Q = \omega_0 \tau/2$

energy ~ $e^{-\omega_0 t/Q}$

modal volume V

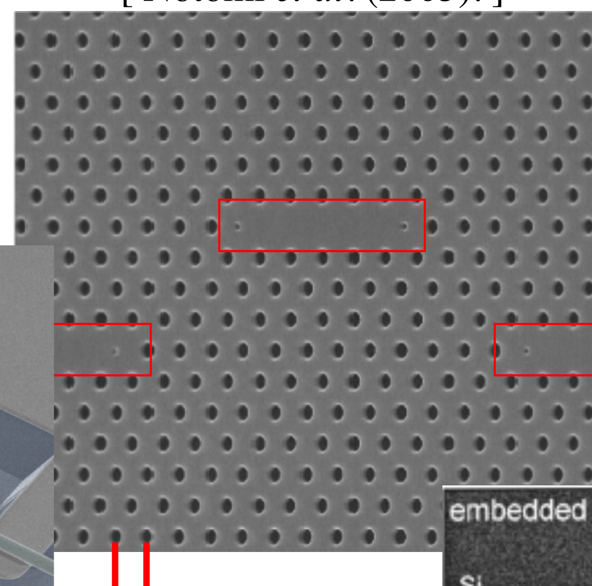


[Schliesser et al., *PRL* **97**, 243905 (2006)]



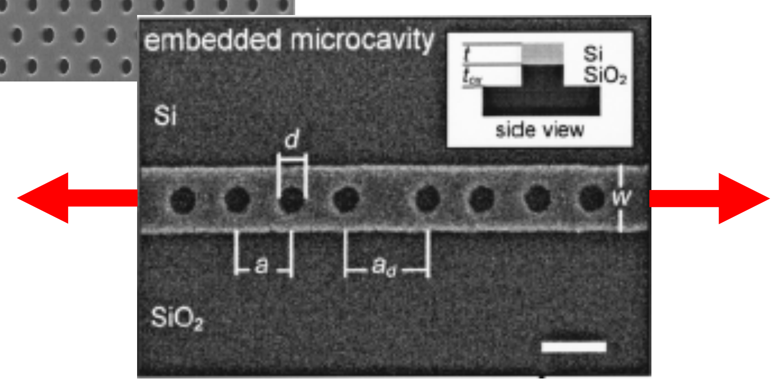
[Eichenfield et al. *Nature Photonics* **1**, 416 (2007)]

[Notomi *et al*. (2005).]



420 nm

[C.-W. Wong, *APL* **84**, 1242 (2004).]



Resonance, Really: Lossless

scalar wave equation

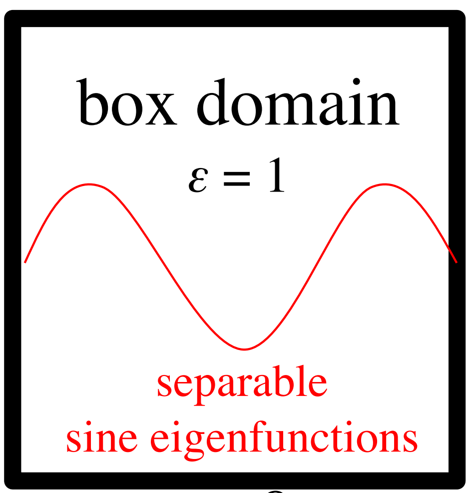
$$\left[\nabla^2 - \varepsilon * \frac{\partial^2}{\partial t^2}\right] u = \text{sources} \xrightarrow{\text{Fourier}} \left[\nabla^2 + \varepsilon(\mathbf{x},\omega)\,\omega^2\right] u = s(\mathbf{x},\omega)$$

scalar Helmholtz equation

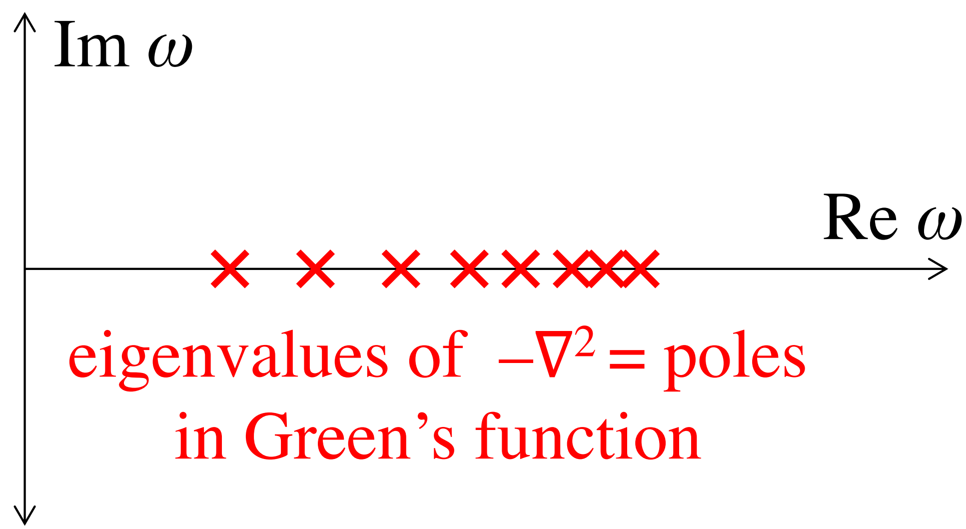
time-harmonic

$u \sim e^{-i\omega t}$

Maxwell “permittivity” ε

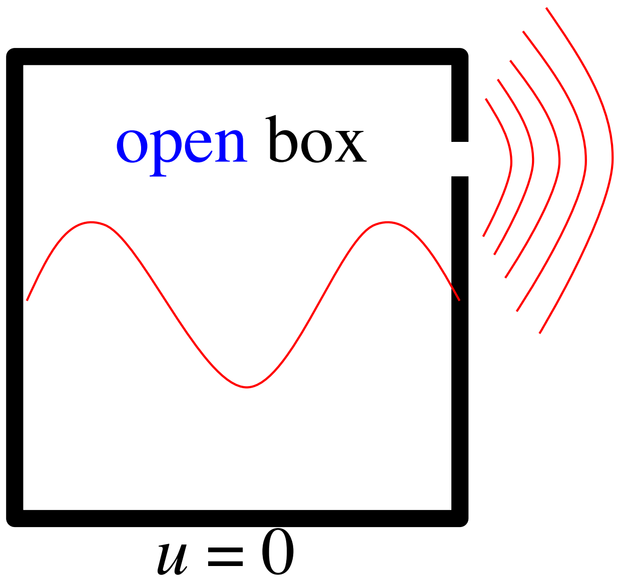


$u = 0$



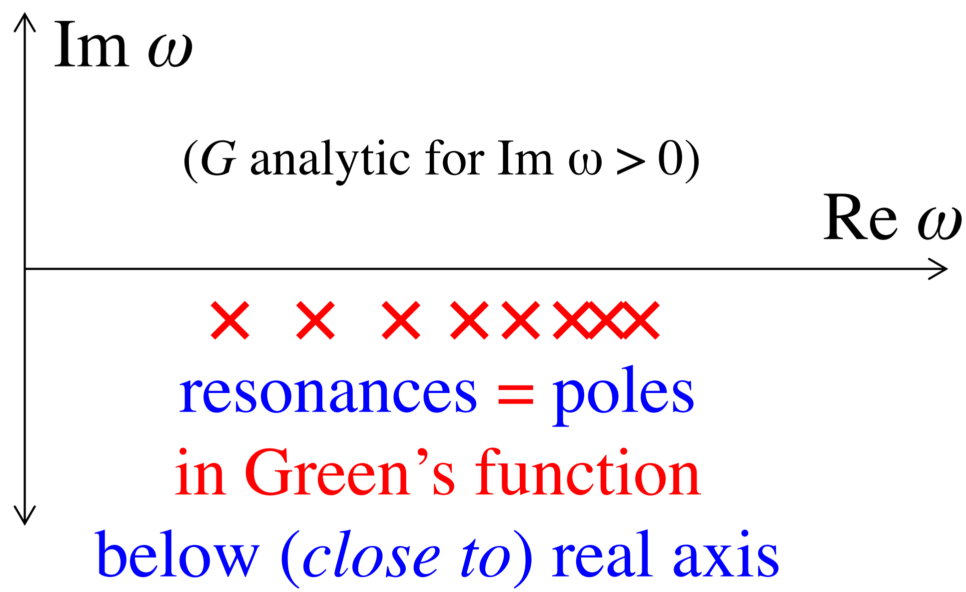
$$\left[\nabla^2 + \varepsilon(\mathbf{x},\omega)\,\omega^2\right] G_\omega(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$$

Resonance, Really: Lossy

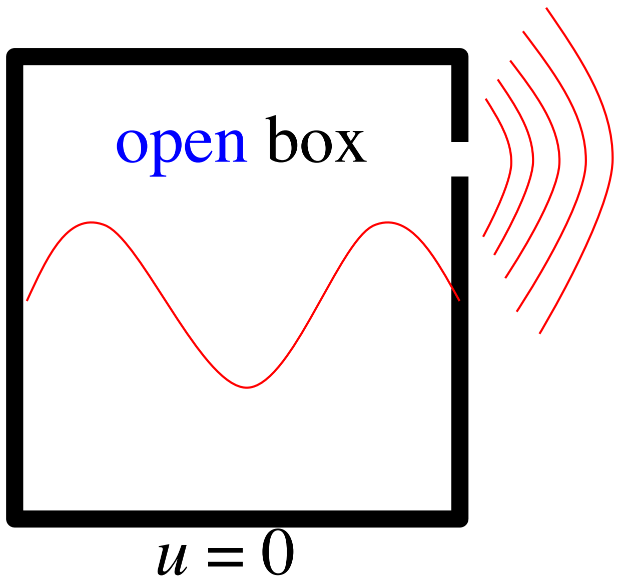


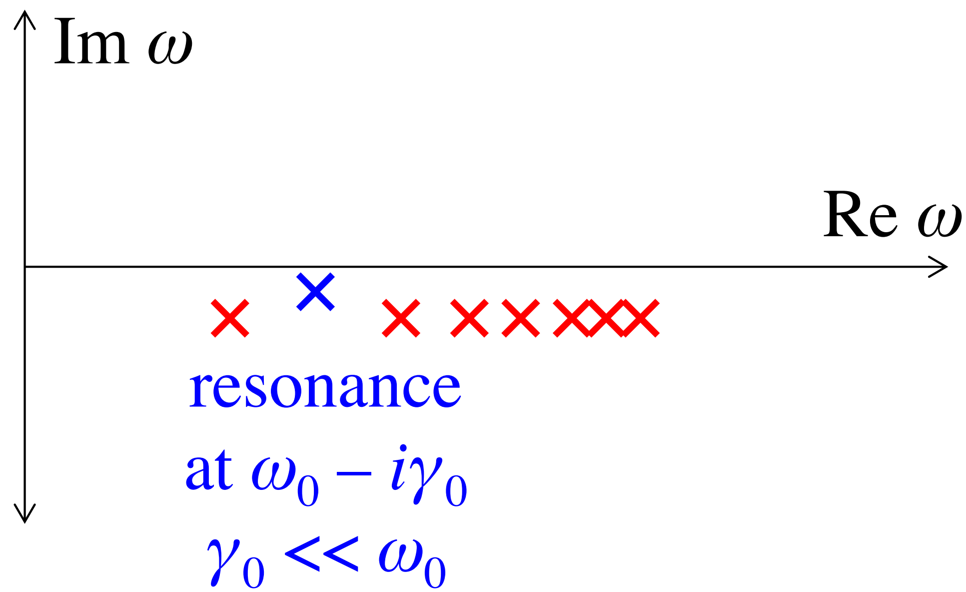
outgoing radiation boundary condition

[= limiting-absorption principle: $\varepsilon + i0^+$]



Resonance, Really: "Leaky Modes"





$$\left[\nabla^2 + \varepsilon(\mathbf{x},\omega)\omega^2\right]u = s(\mathbf{x},\omega)$$

source $s(\mathbf{x},\omega)$ inside box strongly peaked around ω_0

Fourier

local saddle-point approx.
($G \sim$ single pole)

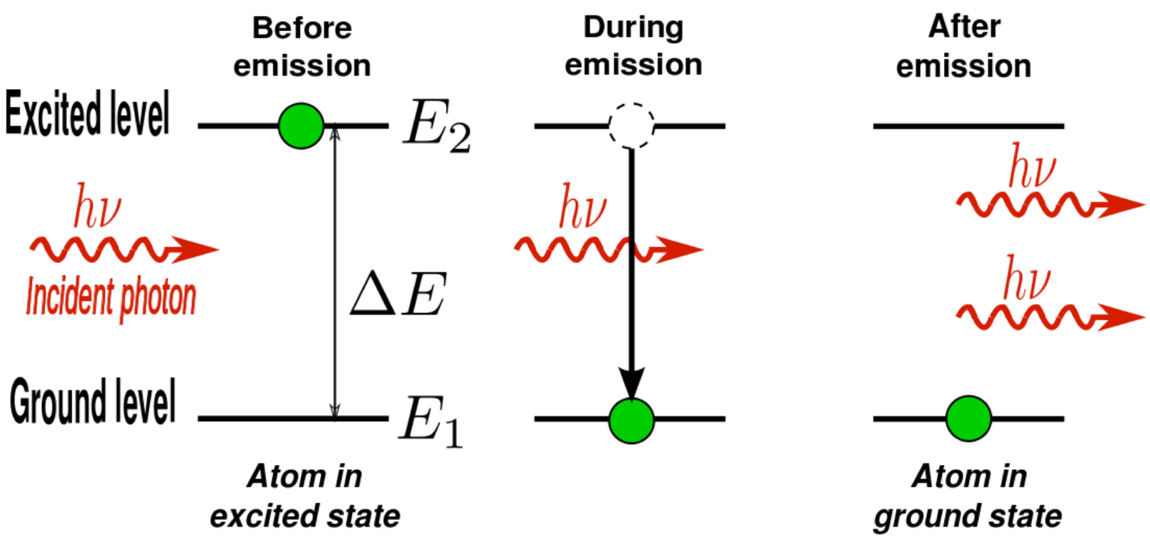
$$u(\mathbf{x},t) \sim e^{-i\omega_0 t - \gamma_0 t}$$

exponentially decaying "leaky mode"

$Q = 2\gamma_0 / \omega_0$

Linear Gain

stimulated emission [wikipedia]



$$E_2 - E_1 = \Delta E = h\nu$$

gain created by
pumping electrons
to population inversion:
= more electrons
in excited state

N_1 = ground state pop.
N_2 = excited state pop.

inversion: $D = N_2 - N_1 > 0$

$u \sim e^{-i\omega t}$ $\quad \left[\nabla^2 + \varepsilon\omega^2\right]u = s(\mathbf{x},\omega)$

gain (exponential *growth* in time):
Im $\varepsilon < 0$ (for $\omega > 0$)

Im $\varepsilon \sim -D$

Nonlinear Gain: Cannot grow forever!

$u \sim e^{-i\omega t}$ $\left[\nabla^2 + \varepsilon\omega^2\right]u = s(\mathbf{x},\omega)$

gain (exponential *growth* in time):
$\operatorname{Im}\varepsilon < 0$ (for $\omega > 0$)

gain created by
pumping electrons
to population inversion:
= many electrons
in excited state

$$\operatorname{Im}\varepsilon \sim -D$$

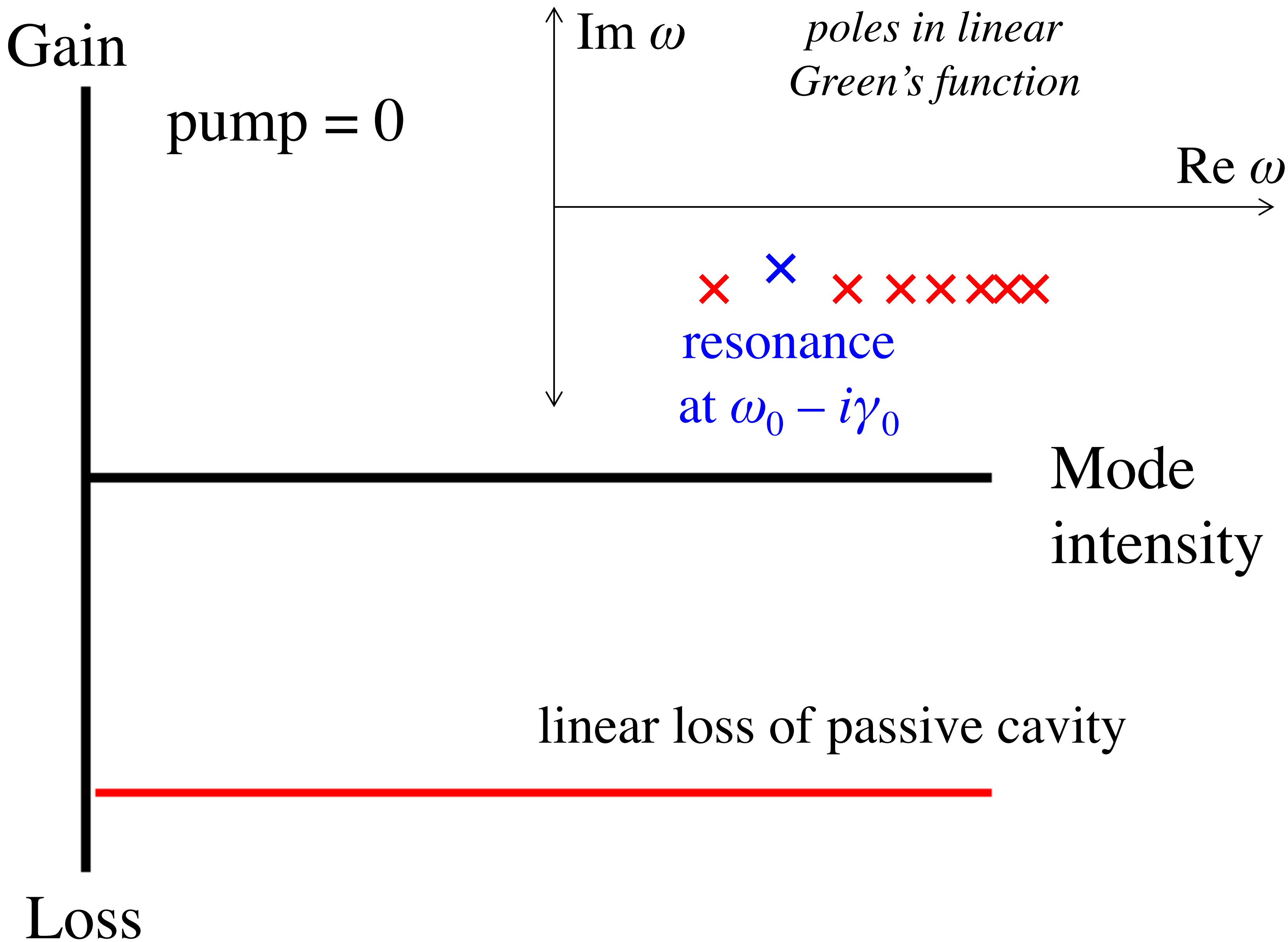
N_1 = ground state pop.
N_2 = excited state pop.

$$D = N_2 - N_1 > 0$$

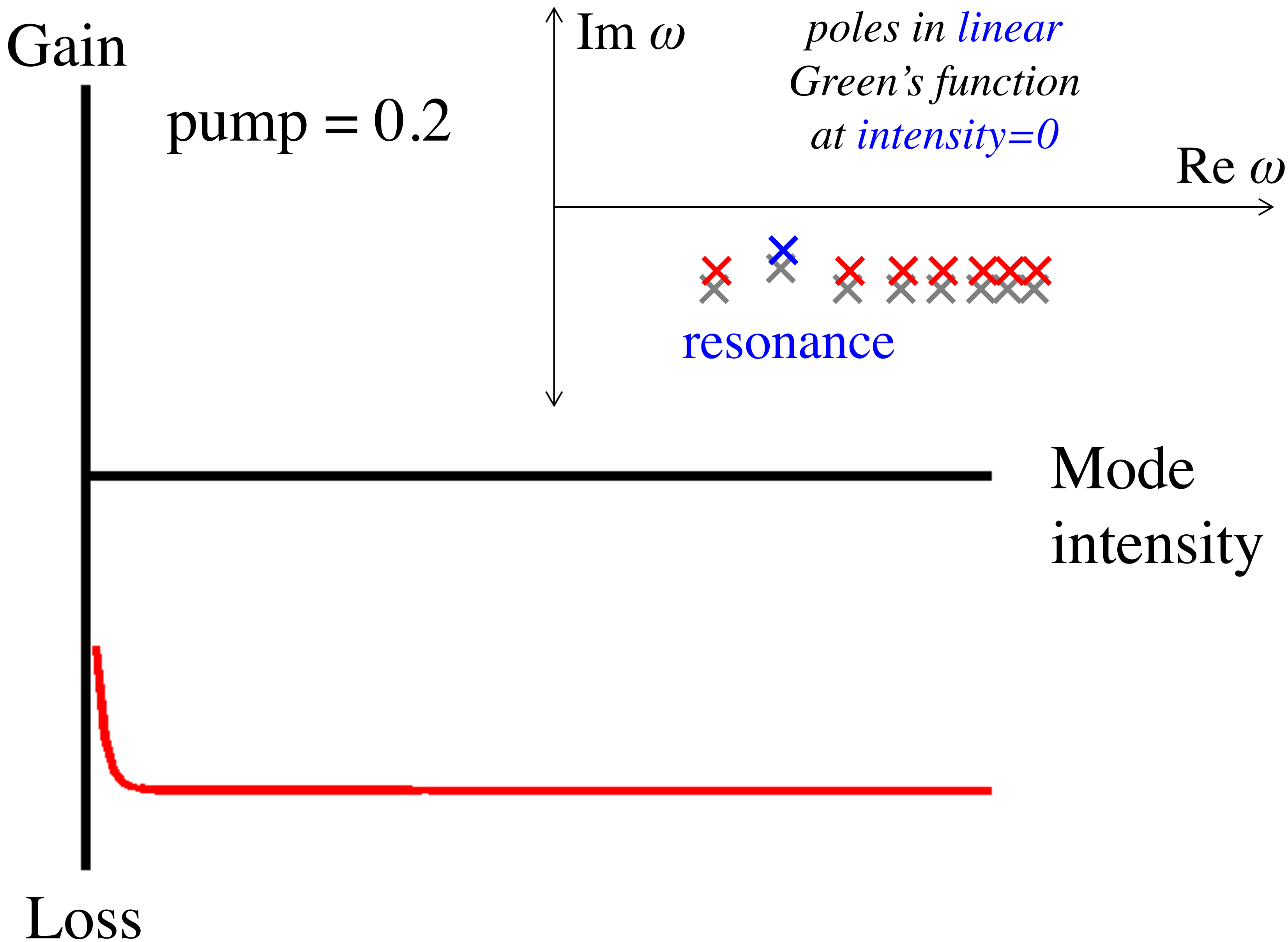
u (electric field) grows exponentially in time… but eventually,
the stimulated emission *depletes* the excited states

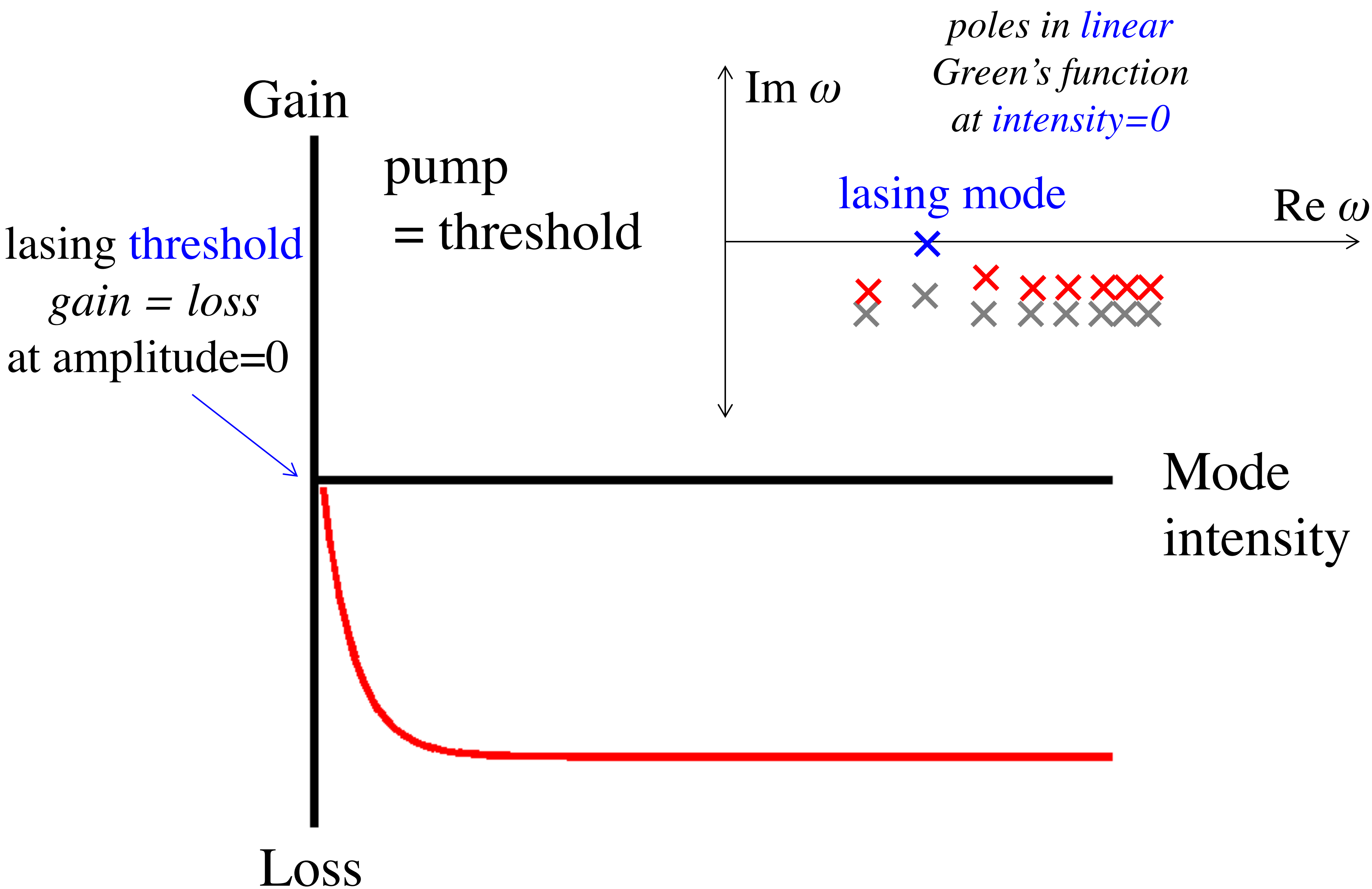
$\Rightarrow$ D decreases with u ($\sim 1/|u|^2$) … "hole burning"

Passive cavity (linear loss)

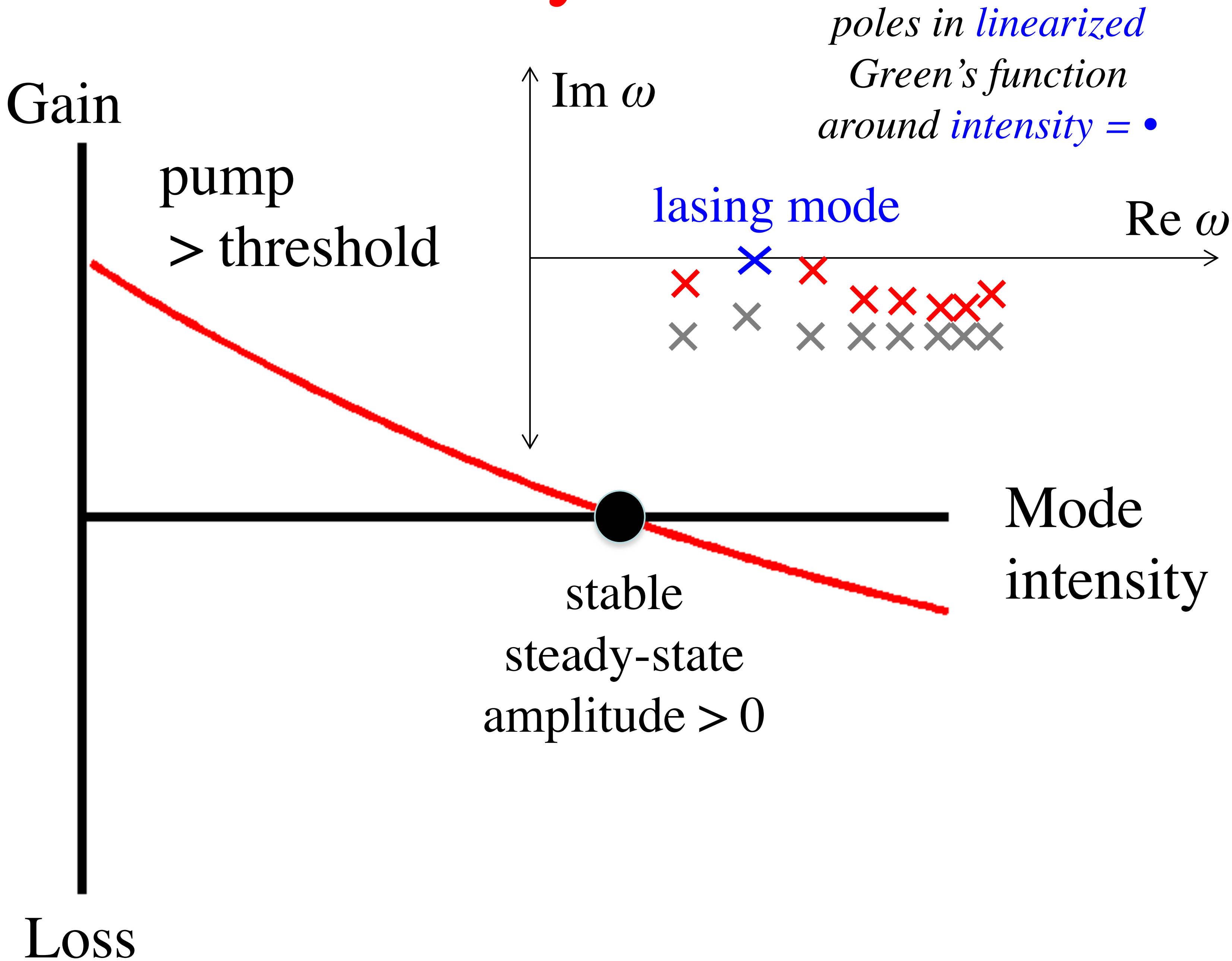


Pump ⇒ Gain: nonlinear in field strength




poles in linear
Green's function
at intensity=0
Im ω
Re ω
lasing mode
Gain
pump
= threshold
lasing threshold
gain = loss
at amplitude=0
Mode
intensity
Loss

The steady state



some goals of laser theory:

for a given laser, determine:

- thresholds
- field emission patterns
- output intensity
- frequencies

of steady-state operation

[if there is a steady state

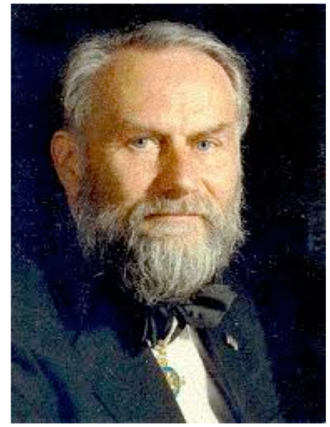
…not true if other resonances too close]

What's new in laser theory

slide:
D. Stone

Lamb

Scully



Haken

Mel Lax

Basic semiclassical theory from early 60's and much of quantum theory

No effective method for accurate solution of the equations for arbitrary resonator including non-linearity, *openness*, multi-mode

Direct numerical solutions in space and time impractical in 3d, hard in 2d

SALT [Tureci, Stone (2006)]: steady-state ab-initio lasing theory

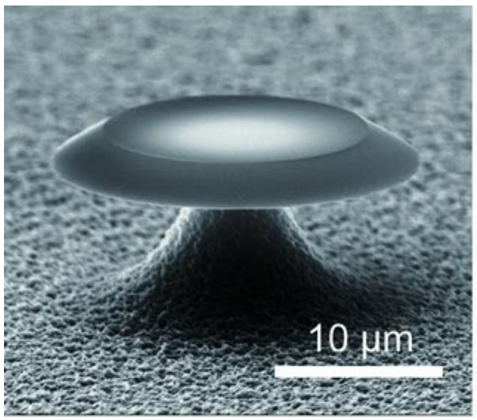
direct solution for the multimode *steady-state* including openness, gain saturation and spatial hole-burning, arbitrary geometry

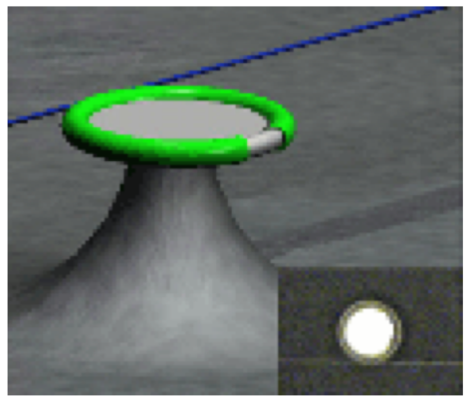
Only inputs are the gain medium … quantitative agreement with brute-force

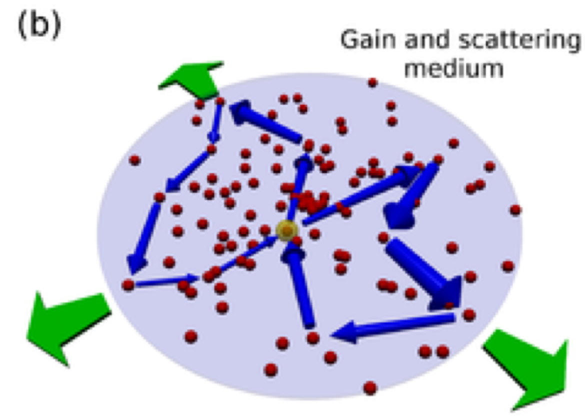
slide:
D. Stone

Motivation: Modern micro/nano lasers

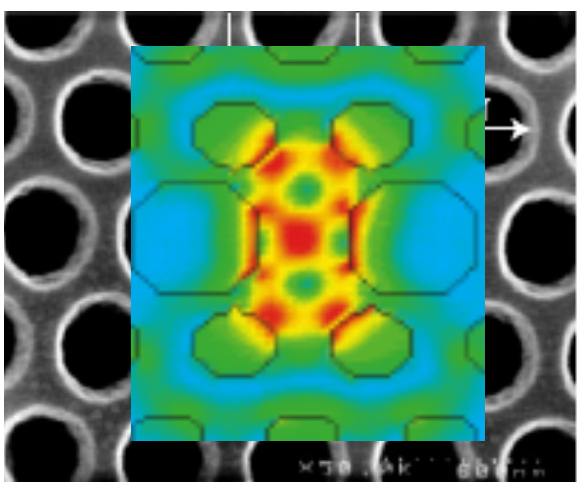
Complex microcavities: micro-disks,micro-toroids, deformed disks (ARCs), PC defect mode, random…

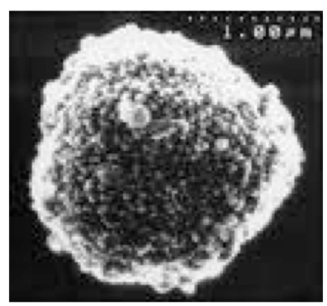




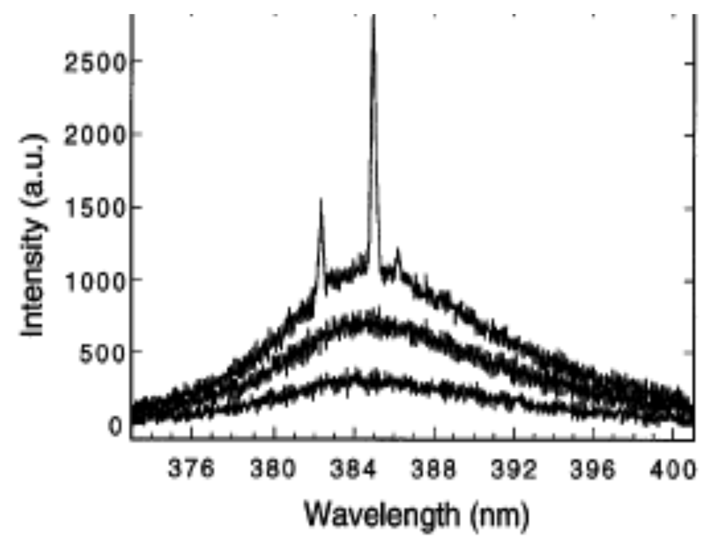


No high-Q passive resonances





No boundary reflection at all!



Semiclassical theory

1. Maxwell equations (classical)

$$-\nabla \times \nabla \times (\mathbf{E}^{+}) - \varepsilon_c \ddot{\mathbf{E}}^{+} = \frac{1}{\varepsilon_0} \ddot{\mathbf{P}}^{+}$$

cavity dielectric

polarization of gain atoms

electric field

$\mathbf{E}^{+} \sim e^{-i\omega t}$

$\mathbf{E} = \mathrm{Re}\, \mathbf{E}^{+}$

Semiclassical theory

1. Maxwell equations (classical)

$$-\nabla \times \nabla \times (\mathbf{E}^{+}) - \varepsilon_c \ddot{\mathbf{E}}^{+} = \frac{1}{\varepsilon_0} \ddot{\mathbf{P}}^{+}$$

cavity dielectric

polarization of two-level gain atoms

2. Damped oscillations of electrons in atoms (quantum)

polarization: $\dot{\mathbf{P}}^{+} = (-i\omega_a - \gamma_{\perp})\mathbf{P}^{+} + \frac{1}{i\hbar}\mathbf{E}^{+} D$

atomic frequency

population inversion (drives oscillation)

Semiclassical theory

1. Maxwell equations

$$-\nabla \times \nabla \times (\mathbf{E}^+) - \varepsilon_c \ddot{\mathbf{E}}^+ = \frac{1}{\varepsilon_0}\ddot{\mathbf{P}}^+$$

cavity dielectric

polarization of two-level gain atoms

2. Damped oscillations of electrons in atoms

$$\dot{\mathbf{P}}^+ = (-i\omega_a - \gamma_\perp)\mathbf{P}^+ + \frac{1}{i\hbar}\mathbf{E}^+ D$$

atomic frequency

population inversion (drives oscillation)

3. Rate equation for population inversion D

$$\dot{D} = \gamma_{\|}(\underbrace{D_0}_{\text{pump}} - D) + \frac{1}{\hbar\omega_a}\mathrm{Re}\left[(\mathbf{E}^+)^\star \cdot \dot{\mathbf{P}}^+\right]$$

rate of work done on "polarization current"

$\gamma_\perp$ and $\gamma_{\|}$ phenomenological relaxation rates (from collisions, etc)

Maxwell–Bloch equations

- fully time-dependent, multiple unknown fields, nonlinear (Haken, Lamb, 1963)

$$-\nabla \times \nabla \times (\mathbf{E}^+) - \varepsilon_c \ddot{\mathbf{E}}^+ = \frac{1}{\varepsilon_0}\ddot{\mathbf{P}}^+$$

Polarization induces inversion

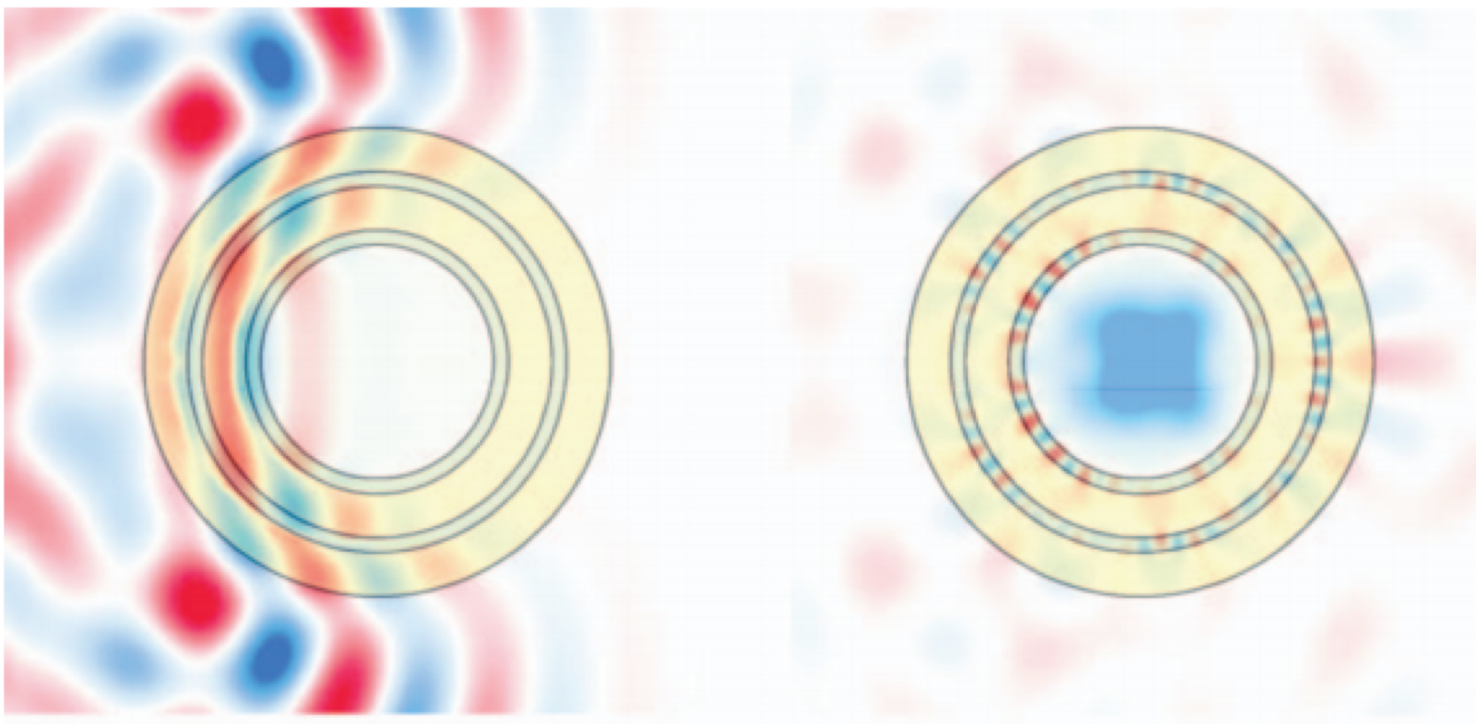
Inversion drives polarization

$$\dot{\mathbf{P}}^+ = (-i\omega_a - \gamma_\perp)\mathbf{P}^+ + \frac{1}{i\hbar}\mathbf{E}^+ D$$

$$\dot{D} = \gamma_{||}(D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^+ \cdot (\mathbf{P}^+)^* - \mathbf{P}^+ \cdot (\mathbf{E}^+)^*]$$

brute-force Maxwell–Bloch

FDTD (finite-difference time-domain) simulations very expensive, but doable



Bermel et. al. (PRB 2006)

Problem: timescales!

$$\gamma_{\|} \ll \gamma_{\perp} \ll \omega_a$$

FDTD takes very long time
to converge to steady state

Solving Maxwell–Bloch for just one set of
lasing parameters is expensive and slow
… supercomputer-scale in 3d …
and systematic design is impractical

Advantage: timescales!

$$\frac{\gamma_{\|}}{\gamma_{\perp}} \ll 1, \quad \frac{\gamma_{\perp}}{\omega_a} \ll 1$$

- hard for numerics
- good for analysis

Ansatz of *M* steady-state modes

$$\mathbf{E}^{+} = \sum_{m=1}^{M} \mathbf{E}_m(\mathbf{x}) e^{-i\omega_m t}$$

$$\mathbf{P}^{+} = \sum_{m=1}^{M} \mathbf{P}_m(\mathbf{x}) e^{-i\omega_m t}$$

…validity checked *a posteriori*

Stationary-inversion approximation

key assumption:

$\gamma_\perp, \Delta\omega \gg \gamma_\parallel$

valid for < 100μm microlasers

- "rotating-wave approximation"

  fast oscillations average out to zero

  … all oscillations are fast compared to $\gamma_\parallel$

$$\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^+ \cdot (\mathbf{P}^+)^* - \mathbf{P}^+ \cdot (\mathbf{E}^+)^*]$$

… leads to:

$$\dot{D} \approx 0$$

stationary-inversion approximation SIE

[neglecting terms ~ fast rates / $\gamma_\parallel$]

before

$$-\nabla\times\nabla\times(\mathbf{E}^+) - \varepsilon_c\ddot{\mathbf{E}}^+ = \frac{1}{\varepsilon_0}\ddot{\mathbf{P}}^+$$

$$\dot{\mathbf{P}}^+ = (-i\omega_a - \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}^+ D$$

$$\dot{D} = \gamma_{||}(D_0 - D) - \frac{2}{i\hbar}[\mathbf{E}^+\cdot(\mathbf{P}^+)^* - \mathbf{P}^+\cdot(\mathbf{E}^+)^*]$$

after:

Steady-State Ab-Initio Lasing Theory,

"SALT"

[Tureci, Stone, 2006]

$$\nabla\times\nabla\times\mathbf{E}_m = \omega_m^2\varepsilon_m\mathbf{E}_m$$

$$\varepsilon_m(\mathbf{x}) = \varepsilon_c(\mathbf{x}) + \frac{\gamma_\perp}{\omega_m - \omega_a + i\gamma_\perp}\left[\frac{D_0(\mathbf{x})}{1+\sum\left|\frac{\gamma_\perp}{\omega_\nu - \omega_a + i\gamma_\perp}\mathbf{E}_\nu\right|^2}\right]$$

Still nontrivial to solve:
equation is nonlinear in both

eigenvalue ω_m ← easier

eigenvector $\mathbf{E}_m$ ← harder

first way to solve SALT: Constant-flux "CF" basis method

Tureci, Stone, PRA 2006
(same paper that introduced SALT)

$$\mathbf{E}_m(\mathbf{x}) = \sum_{n=1}^{N} c_{mn}\mathbf{F}_n(\mathbf{x})$$

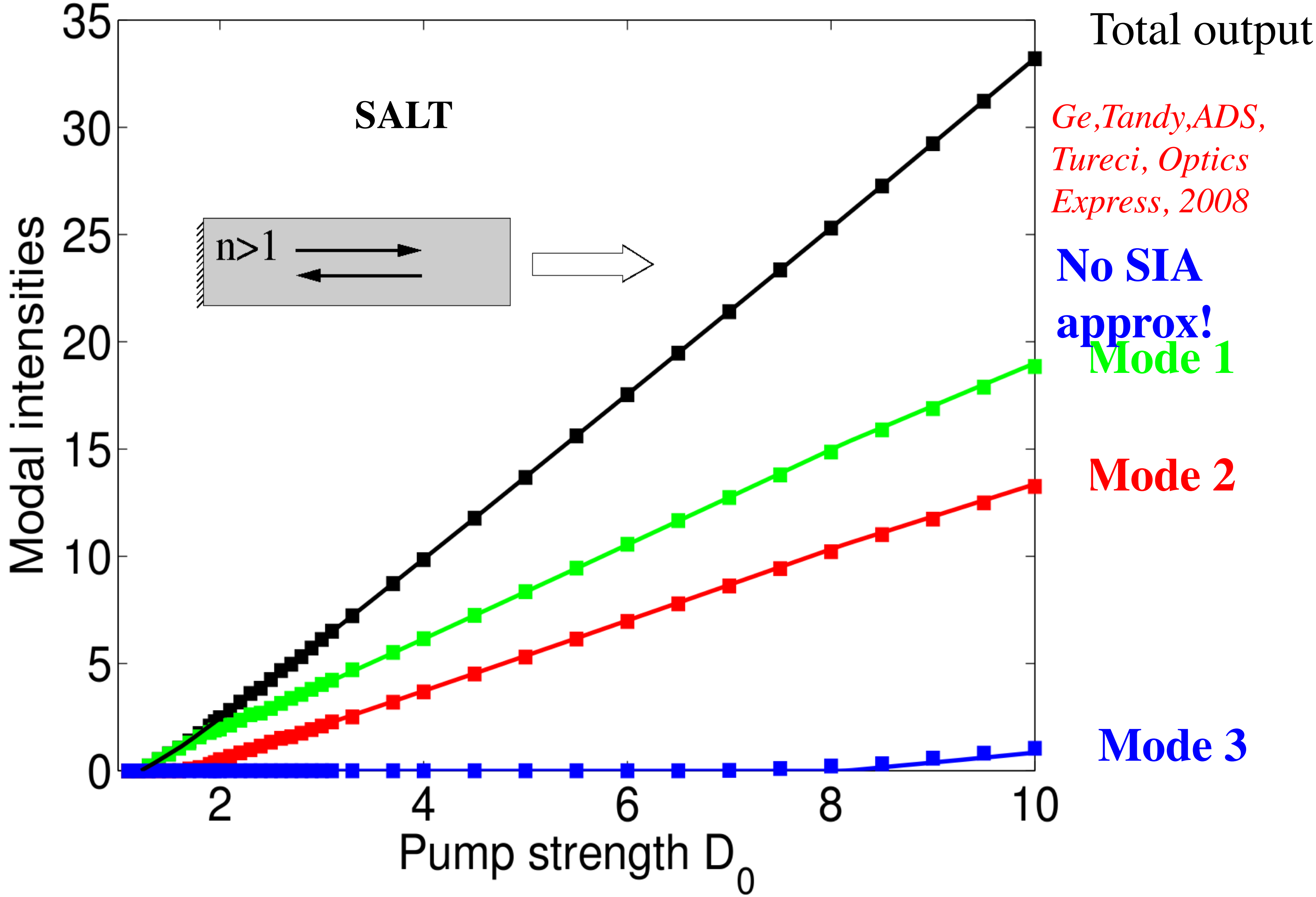
solutions to *linear* problem at threshold

$$\mathbb{T}(\omega_m, c_{mn})c_{mn} = 0$$

problem still nonlinear, but
very small dimensionality

Comparison of SALT and Maxwell-Bloch: intensities

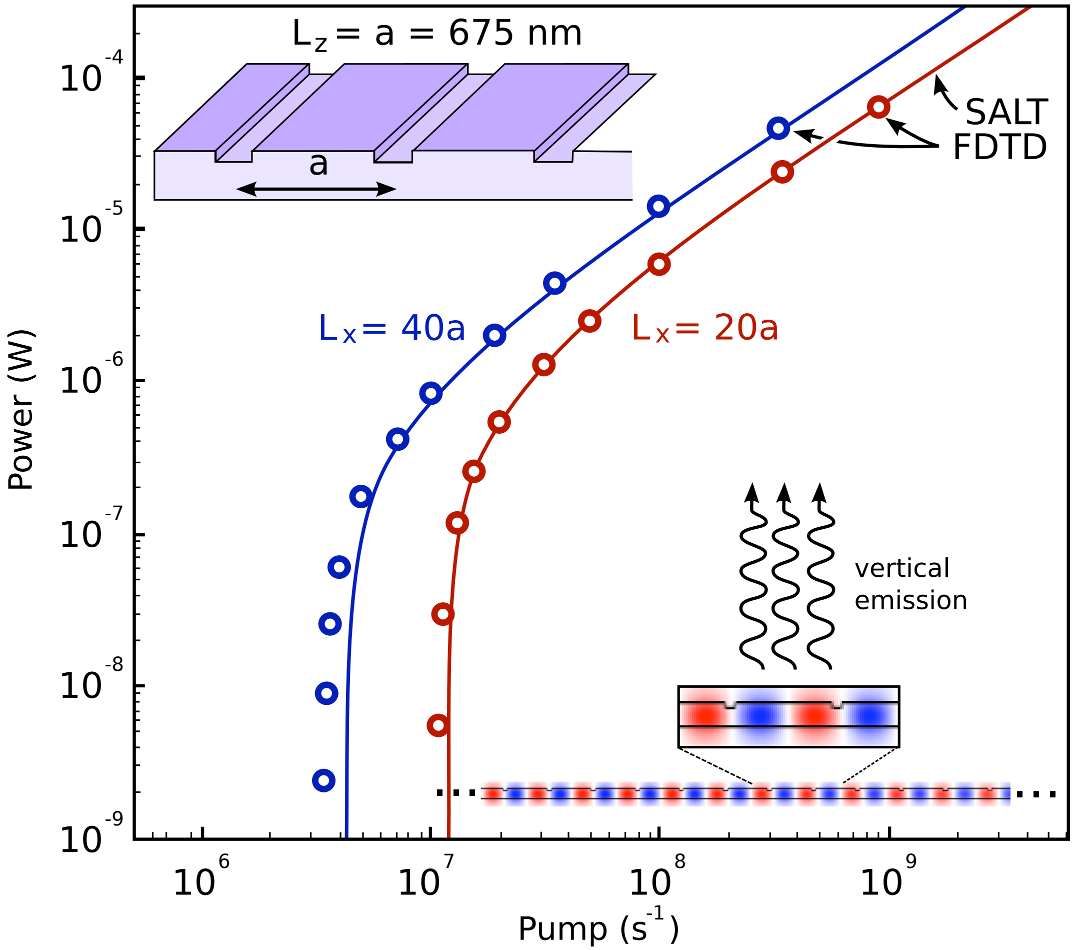
slide: D. Stone



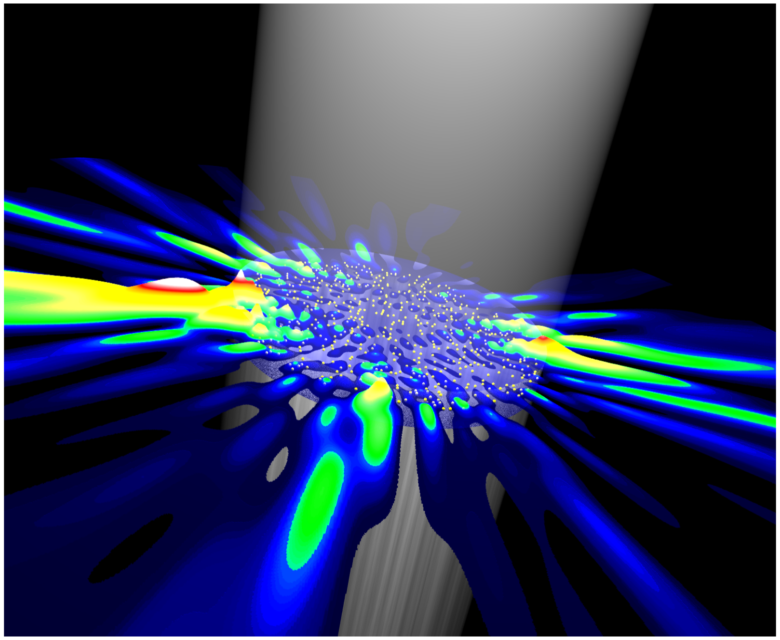
“Realistic” application to novel lasers

slide: D. Stone

Chua, Chong, ADS, Soljacic, Bravo-Abad, Opt. Express 2010



2D Random Lasers



“Strong interactions in multimode random lasers”, H. Tureci, L. Ge, S. Rotter, ADS; Science, 320,643 (2008) – random lasing is “conventional”

CF basis method not scalable

1. far above threshold, expansion efficiency decreases, need more basis functions
2. in most cases basis functions need to be obtained numerically
3. huge basis = huge storage, time in 2d and 3d

Common pattern for theoretical models

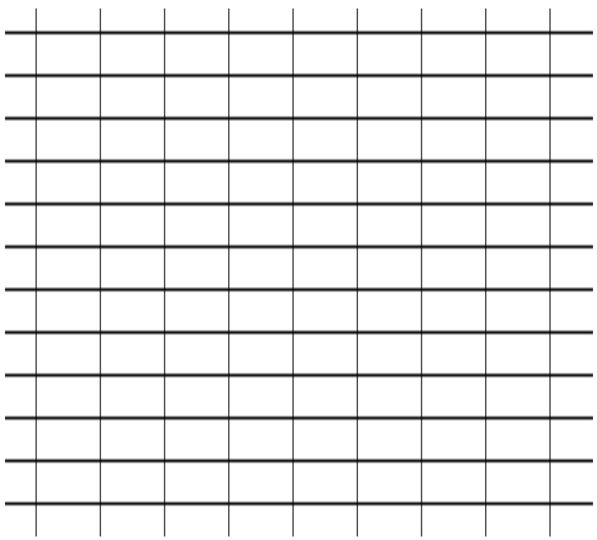
1. purely analytical solutions (handful of cases)
2. specialized basis (problem-dependent and hard to scale to arbitrary systems)
3. generic grid/mesh, discretize

SALT was here

Can we solve the equations of SALT (which are nonlinear) on a grid **without an intermediate basis?**

Finite-difference discretization (FDFD)

$$\nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m(\omega_m, \{\mathbf{E}_\nu\})\mathbf{E}_m$$



degrees of freedom:

$\mathbf{E}_m$ at every point on (Yee) grid

m = 1,2,…# modes

$\nabla \times \nabla \times$ → finite differences

→"just" solve

… but is it reasonable to solve 10^4–10^7 coupled nonlinear equations?

Yes!

Newton: $\mathbf{f}(\mathbf{v}) = 0$ $\qquad \mathbf{v}_{\text{guess}} \to \mathbf{v}_{\text{guess}} - \left(\frac{\partial \mathbf{f}}{\partial \mathbf{v}}\right)^{-1} \mathbf{f}$

$$\mathbf{v} = \begin{pmatrix} \mathbf{E}_m \\ \omega_m \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \left[-\nabla \times \nabla \times + \omega_m^2 \varepsilon_m\right] \mathbf{E}_m \\ \mathbf{E}_m(\mathbf{x}_0) \end{pmatrix}$$

key fact #1:

Newton's method converges very quickly when we have a good initial guess (near the actual answer)

key fact #2:

we *have* a good initial guess: at threshold, the problem is linear in $\mathbf{E}_m$, easy to solve)

(omitted details)

… sparse solvers for Newton steps

… eliminating spurious $\mathbf{E}_m=0$ solutions

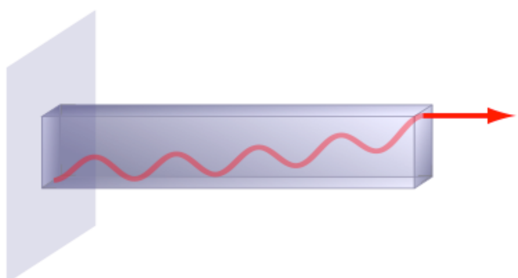
… linear solvers for passive modes (Im $\omega < 0$ poles)

… watch for thresholds of additional modes

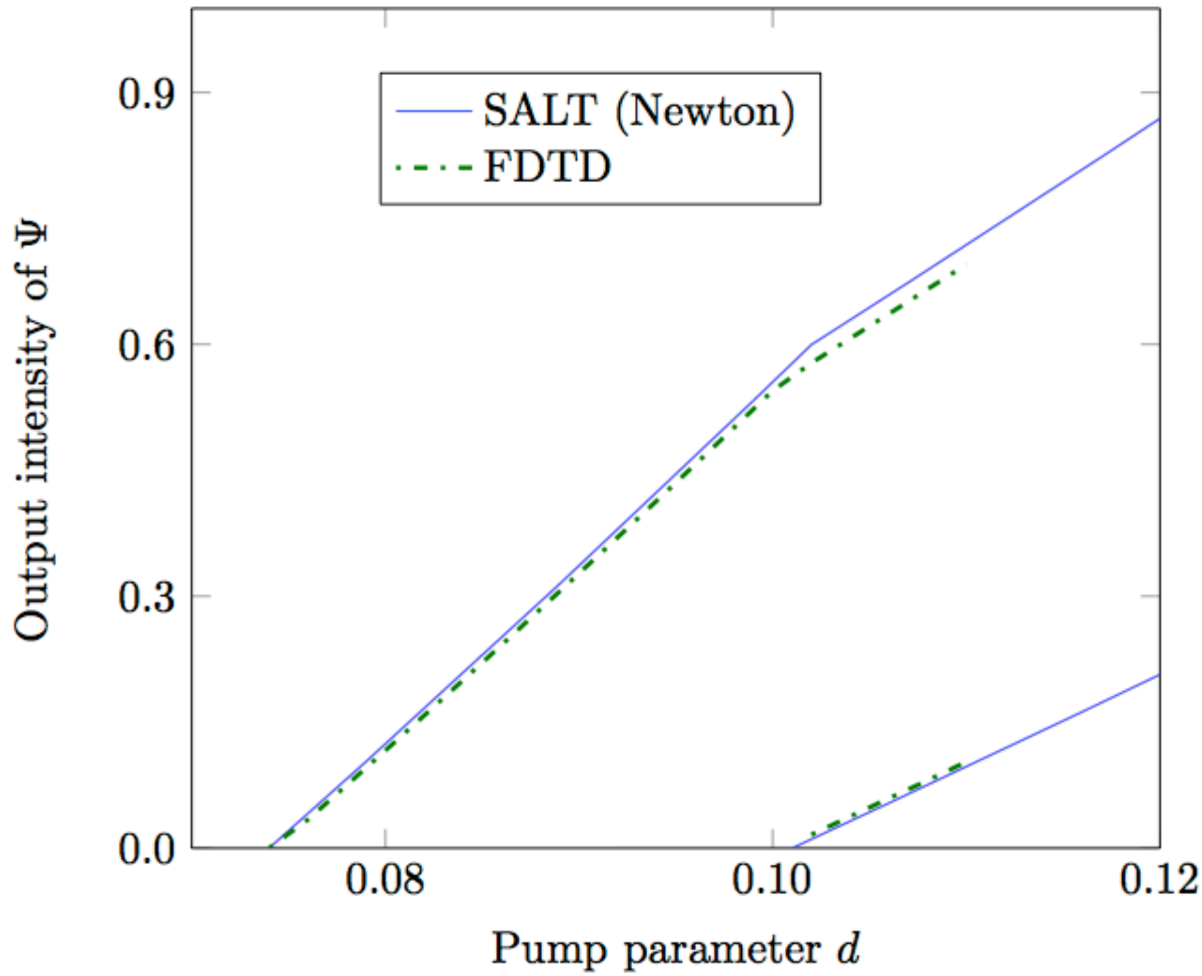
… *a posteriori* stability check

Benchmark comparison with previous 1d results

1d laser cavity



Benchmarks for ~1000 pixels
Maxwell—Bloch (FDTD)
~60 CPU hours
SALT, Direct Newton
20 CPU seconds!!!

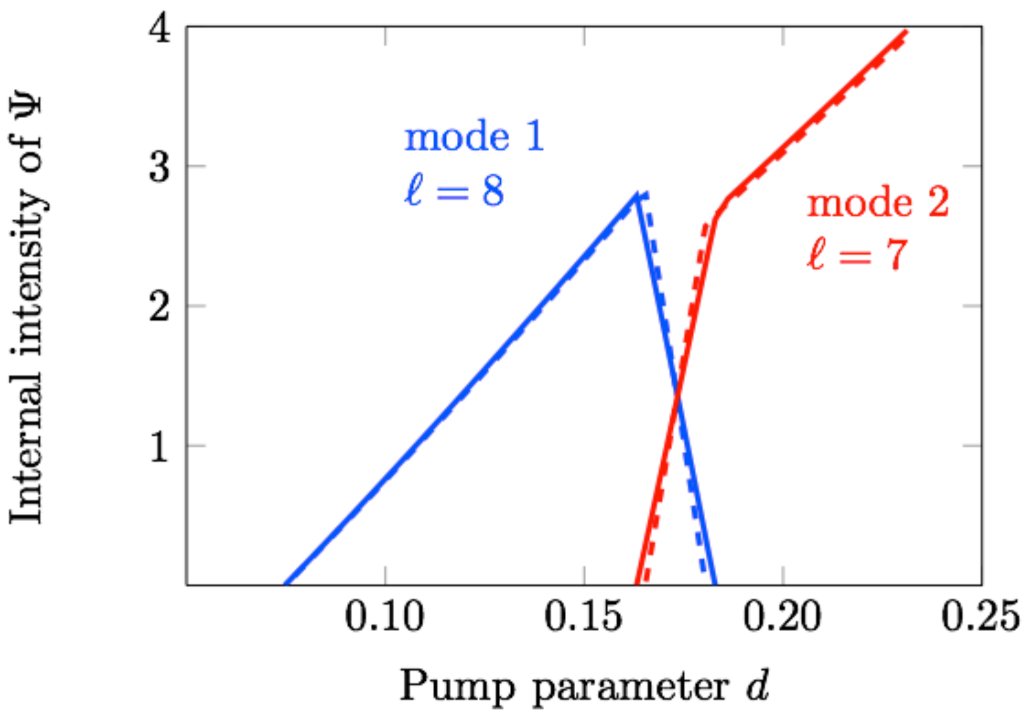


c.f. SALT CF Basis
~5 CPU minutes

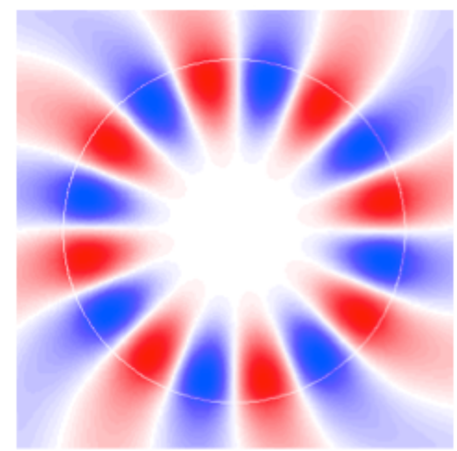
… much easier to optimize simple FDFD code!

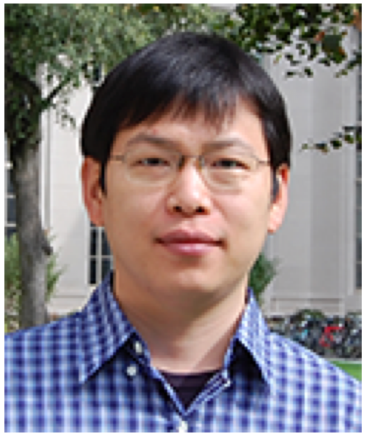
Mode-switching lasers in 2d

mode-switching behavior in microdisk laser (solid = Newton, dotted = Bessel basis)



field profile of mode 1





From Newton to Anderson

[Wonseok Shin et al, manuscript in preparation (2018)]

Problem: Newton's method requires you to rip your existing optimized Maxwell solver to shreds and re-assemble it into the SALT Jacobian matrix

… $|E|^2$ terms mean you need to write in terms of real matrices of real/imaginary parts

Solution: combine an existing ω-domain *linear* Ax=b (Maxwell/Helmholtz) solver with Anderson acceleration (1965) of a carefully chosen fixed-point equation f(x) = x

[Walker & Ni (2011): essentially ~ GMRES Newton]

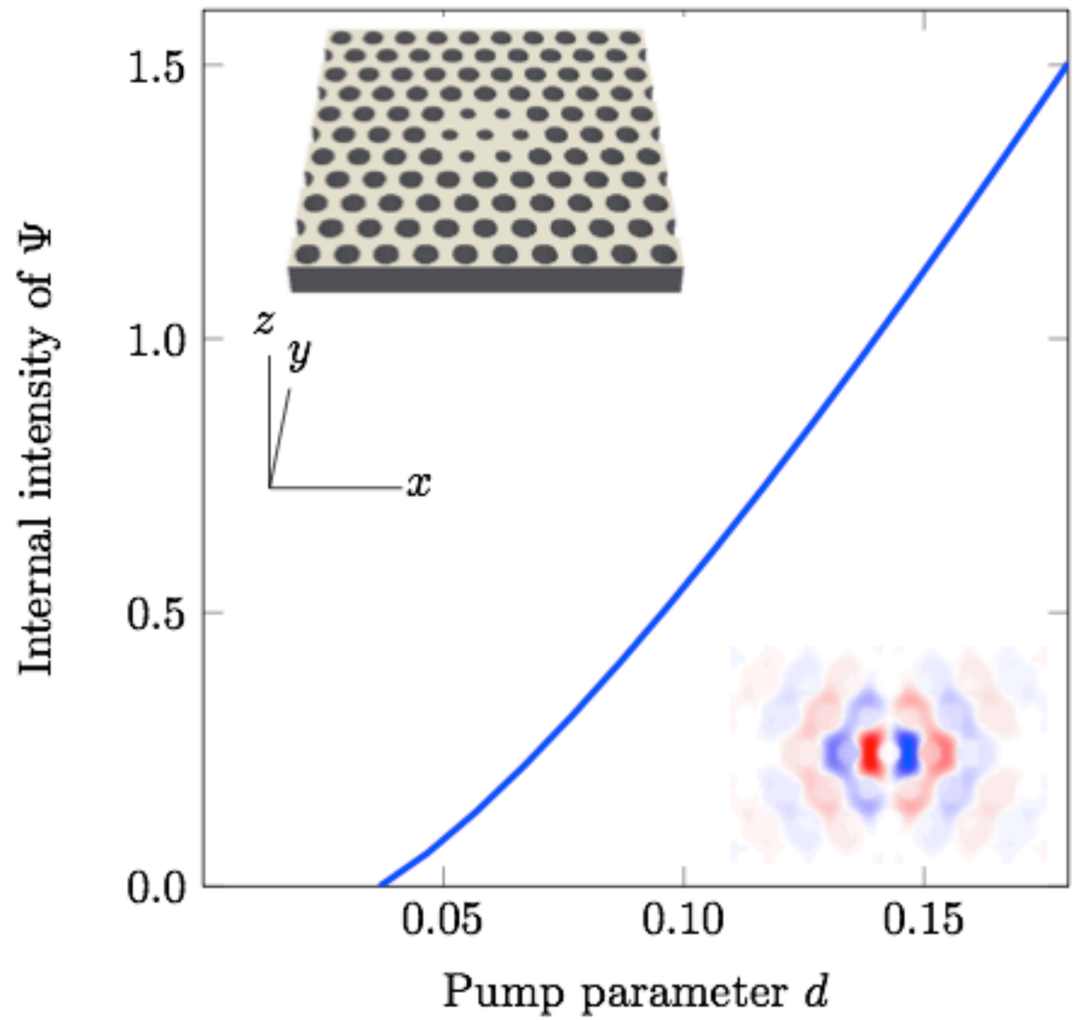
= black-box linear solver + derivative-free updates for the nonlinearity

~ 2–3x more iterations than Newton (10–30 vs. 5–10).

Full 3d calculation

full-vector simulation of lasing defect mode in photonic crystal slab

~ 50 x 50 x 30 pixel computational cell:
10 CPU minutes on a laptop with SALT + Newton's method

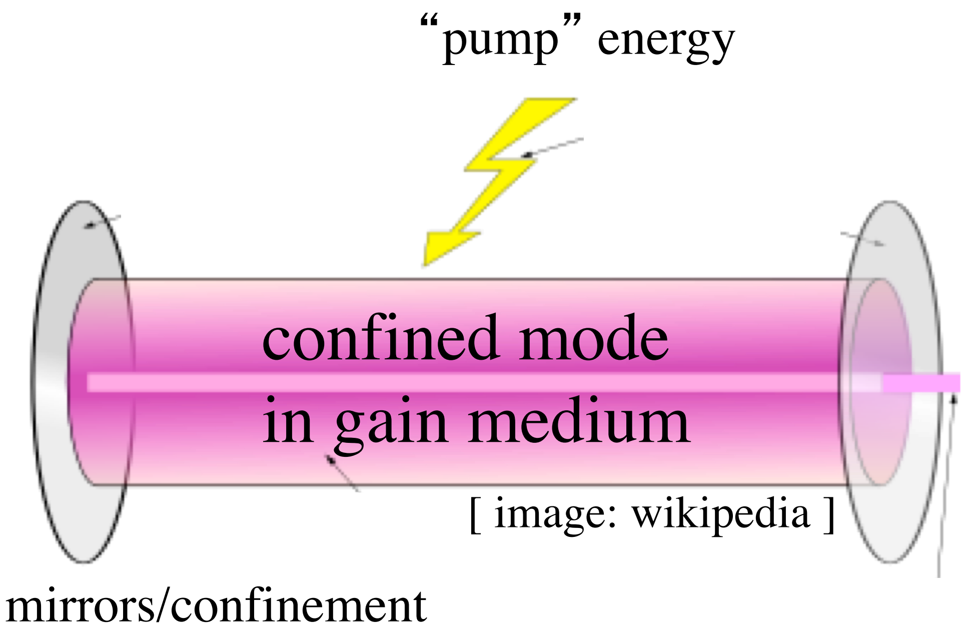


Today's menu

- Laser basics
- The SALT nonlinear eigenproblem
- Noise, linear-response theory, & linewidths

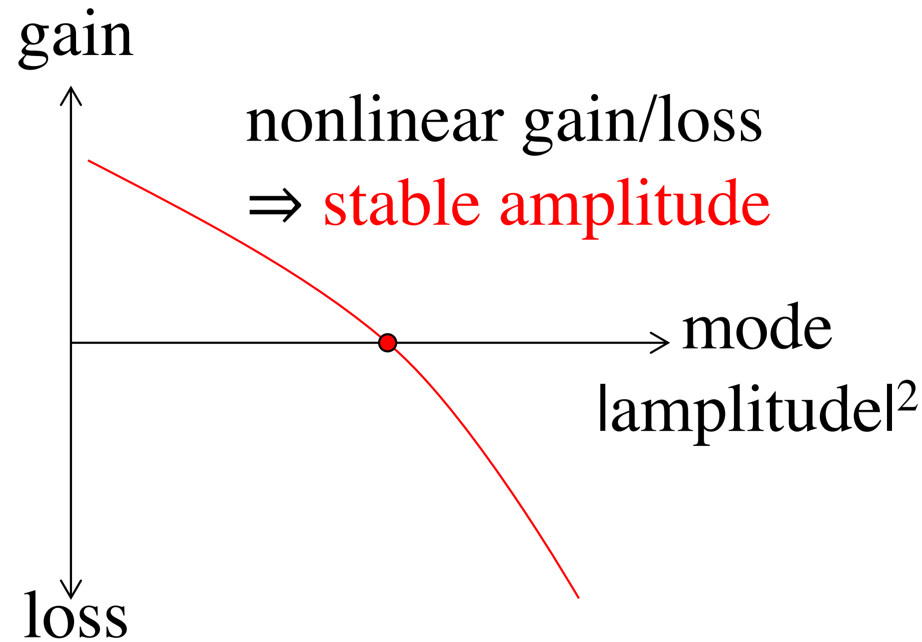
Lasers: Quick Review

laser = lossy optical resonance + nonlinear gain



threshold: increase pump until
gain ≥ loss at amplitude=0

above threshold:



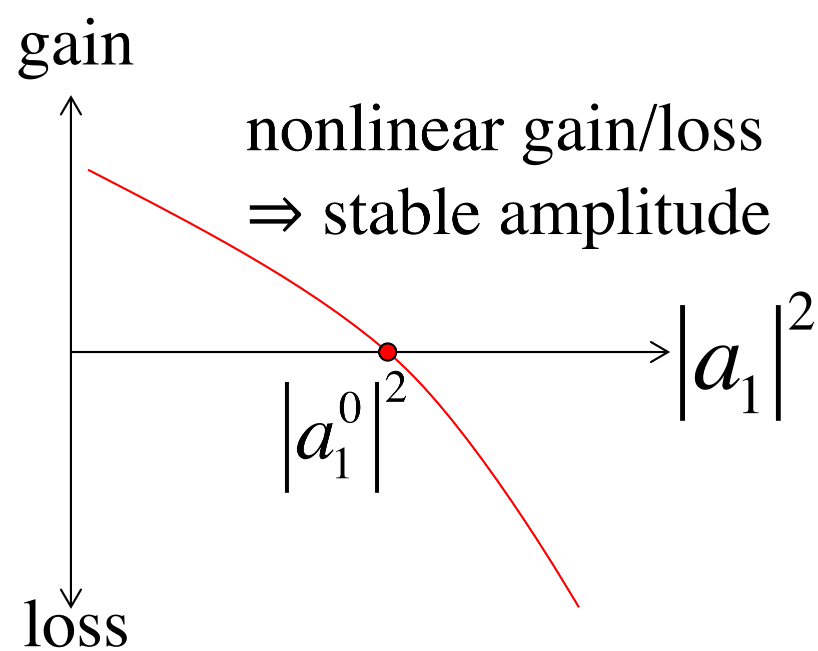
Lasers: Quick Review

laser = lossy optical resonance + nonlinear gain

toy "van der Pol" oscillator model of single-mode laser [e.g. Lax (1967)]:

$$a_1(t)\mathbf{E}_1(\mathbf{x})e^{-i\omega_1 t}$$

above threshold:



(toy instantaneous nonlinearity)

$$\frac{da_1}{dt} = C_{11}\left(\left|a_1^0\right|^2 - \left|a_1\right|^2\right)a_1 \Rightarrow a_1 \to a_1^0$$

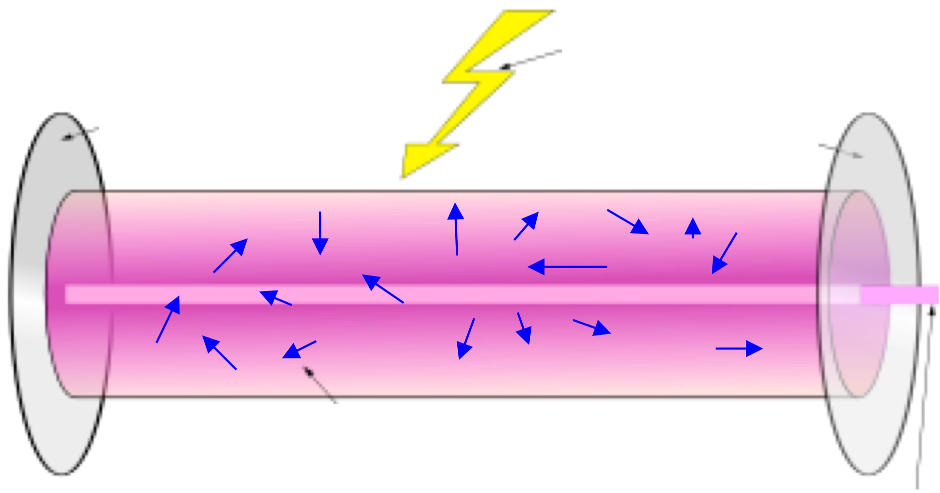
nonlinear coefficient

steady state

= zero linewidth!

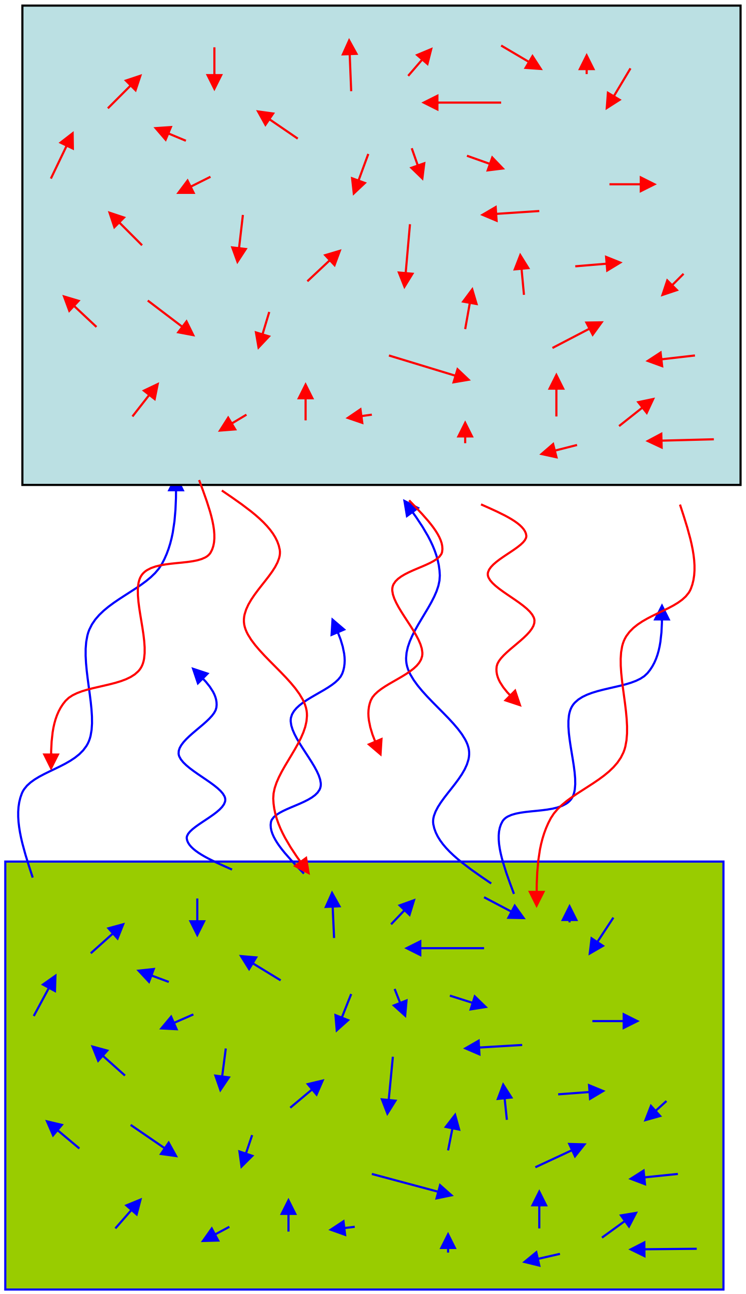
(δ-function spectrum)

Laser noise:



random (quantum/thermal) currents
"kick" the laser mode
⇒ Brownian phase drift = finite linewidth

Microscopic current fluctuations



Fluctuating currents **J** produce fluctuating electromagnetic fields.

Fields carry:

- Momentum ⇒ Casimir forces
- Energy ⇒ thermal radiation

In a laser: **J** = random forcing
= phase drift
= nonzero laser linewidth

Toy Laser + Noise

[= nonlinear “van der Pol” oscillator,
similar to e.g. Lax (1967)]

lowest-order stochastic ODE:

$$\frac{da_1}{dt} \approx C_{11}\left(\left|a_1^0\right|^2 - \left|a_1\right|^2\right)a_1 + f_1(t)$$

random forcing

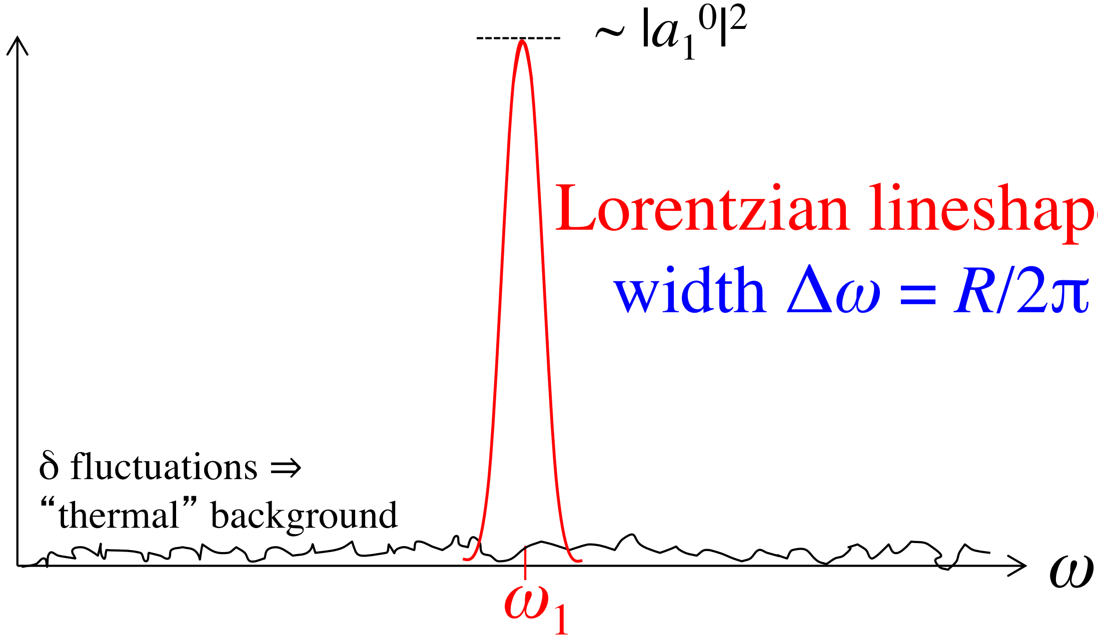
tricky part: getting f & C

linearize:

$$a_1 = \left[a_1^0 + \delta_1(t)\right]e^{i\varphi_1(t)}$$

$$\Rightarrow \ldots \Rightarrow \langle\phi^2\rangle = Rt$$

Brownian (Wiener) phase



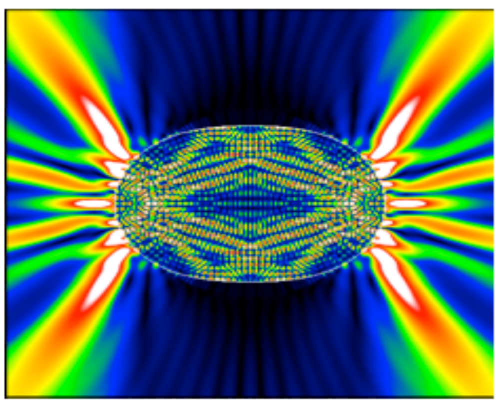
Linewidth formulas: a long history

$$\Gamma = \frac{\hbar\omega_0\gamma_c^2}{2P} \cdot \frac{N_2}{N_2 - N_1} \cdot \left| \frac{\int_C dx |\mathbf{E}_c|^2}{\int_C dx \mathbf{E}_c^2} \right|^2 \cdot \left(\frac{\gamma_\perp}{\gamma_\perp + \frac{\gamma_c}{2}} \right)^2 \cdot (1 + \alpha^2)$$

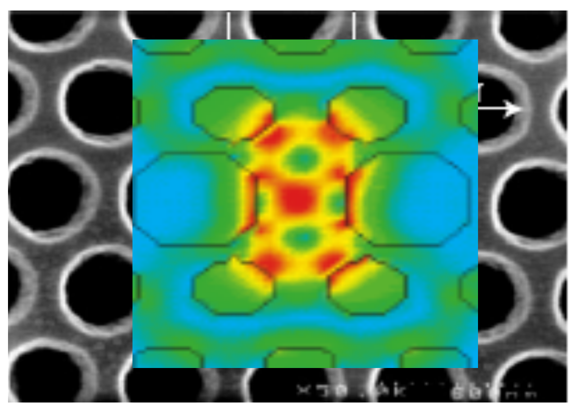
ST **I** **P** **B** **α**

- **S**chawlow-**T**ownes ('58) - inverse power 1/P scaling
- **I**ncomplete inversion ('67) - due to partial inversion
- **P**etermann ('79) - enhancement for lossy cavities
- **B**ad-cavity ('67) - reduction due to dispersion
- **α**-factor ('82) - coupling of intensity/phase fluctuations

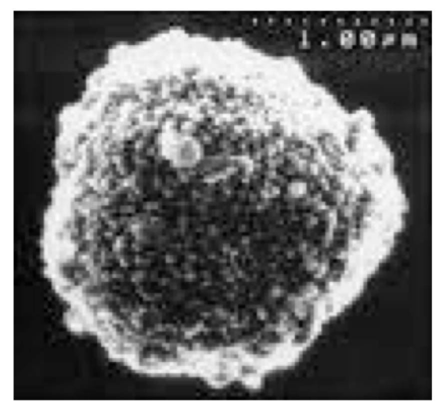
… all make approximations invalid for *μ*-scale lasers…



chaotic cavity



photonic crystal



random laser

Starting point:

Maxwell–Bloch

electric field $$\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2}\ddot{\mathbf{E}} = \frac{4\pi}{c^2}\left[\ddot{\mathbf{P}}^+ + (\ddot{\mathbf{P}}^+)^*\right]$$

gain polarization $$\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}D$$

population inversion $$\dot{D} = \gamma_\parallel(D_0 - D) - \frac{2}{i\hbar}\mathbf{E}\cdot\left[(\mathbf{P}^*)^+ - \mathbf{P}^+\right]$$

[Arecchi & Bonifacio, 1965]

Starting point:

Langevin Maxwell–Bloch

electric field $$\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2}\ddot{\mathbf{E}} = \frac{4\pi}{c^2}\left[\ddot{\mathbf{P}}^+ + (\ddot{\mathbf{P}}^+)^*\right] - \frac{4\pi}{c}\dot{\mathbf{J}}$$

noise

gain polarization $$\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar}\mathbf{E}D$$

population inversion $$\dot{D} = \gamma_\parallel(D_0 - D) - \frac{2}{i\hbar}\mathbf{E}\cdot\left[(\mathbf{P}^*)^+ - \mathbf{P}^+\right]$$

[Arecchi & Bonifacio, 1965]

Noise correlations: fluctuation–dissipation theorem at $T < 0$

$$\langle J_i(\omega,x)J_j^*(\omega,x')\rangle = \frac{\omega}{\pi}\delta_{ij}\delta(x-x')\left[\frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2kT}\right)\right]\operatorname{Im}\varepsilon(x)$$

[Callen & Welton, 1957]

The **N**oisy-SALT linewidth

[Pick et al., PRA **91**, 063806 (2015)]

Starting point:
Langevin MB.
(with **SALT** + **FDT**)

Maxwell
perturbation theory

Dynamical eqs.
for lasing mode
amplitudes
(**oscillator eqs.**)

ODE linearization +
closed-form
integration

formulas for
multimode
linewidths &
RO side peaks

Oscillator equations

Noise-free **SALT:** $\mathbf{E}(\mathbf{x},t) = \sum_\mu \mathbf{E}_\mu(\mathbf{x}) a_{\mu 0} e^{-i\omega_\mu t}$

SALT modes

Noisy **N-SALT:** $\mathbf{E}(\mathbf{x},t) = \sum_\mu \mathbf{E}_\mu(\mathbf{x}) a_\mu(t) e^{-i\omega_\mu t}$

Simple limit: Single-mode "class A" lasers

$$\frac{da_1}{dt} = \underbrace{C_{11}\left(a_{10}^2 - |a_1|^2\right) a_1}_{\text{instantaneous restoring force}} + f_1$$

often derived heuristically [Lax (1967)]

Most general dynamical equations (class A+B lasers)

$$\dot{a}_\mu = \sum_\nu \underbrace{\left[\int dx\, c_{\mu\nu}(x)\, \gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} \left(a_{\nu 0}^2 - |a_\nu(t')|^2\right)\right] a_\mu}_{\text{time-delayed, spatially inhomogeneous restoring force}} + f_\mu$$

Solving the oscillator equations

$$\dot{a}_\mu = \sum_\nu \left[\int dx\, c_{\mu\nu}(x)\, \gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} \left(a_{\nu 0}^2 - |a_\nu(t')|^2 \right) \right] a_\mu + f_\mu$$

Expand mode amplitudes around steady state:
$a_\mu = (a_{\mu 0} + \delta_\mu) \exp(i\varphi_\mu)$ [small noise = linearize in δ_μ]

○**Miracle #1:** can solve analytically for $< \varphi_\mu \varphi_\nu >$ correlation function, which gives linewidths.

○**Miracle #2:** $\gamma(x)$ exactly cancels and gives same answer as instantaneous model! The simple "class A" model is correct for "class B!"

Single-mode linewidth formula

[Pick et al., PRA **91**, 063806 (2015)]

cavity bandwith

$$\left| \frac{\int dx\,(\omega_0 \mathrm{Im}\,\varepsilon)\mathbf{E_0}^2}{\int dx\,\varepsilon\,\mathbf{E_0}^2} \right|$$

Petermann factor

$$\left| \frac{\int_{\mathrm{P}} dx\,\mathrm{Im}\,\varepsilon\,|\mathbf{E}_0|^2}{\int dx\,\mathrm{Im}\,\varepsilon\,\mathbf{E_0}^2} \right|^2$$

Bad-cavity factor

$$\left| \frac{\int dx\,\varepsilon\,\mathbf{E}_0^2}{\int dx\,\mathbf{E}_0^2\left(\varepsilon + \frac{\omega_0}{2}\frac{\partial\varepsilon}{\partial\omega_0}\right)} \right|^2$$

$$\Gamma = \frac{\hbar\omega_0\widetilde{\gamma}_0^2}{2P} \cdot \widetilde{n}_{\mathrm{sp}} \cdot \widetilde{K} \cdot \widetilde{B} \cdot (1+\widetilde{\alpha}^2)$$

ST **I** **P** **B** **α**

$$\frac{\int dx\left[\frac{1}{2}\coth(\frac{\hbar\omega\beta}{2}) - \frac{1}{2}\right]\mathrm{Im}\varepsilon|\mathbf{E}_0|^2}{\int_{\mathrm{P}} dx\,\mathrm{Im}\,\varepsilon\,|\mathbf{E}_0|^2}$$

Incomplete inversion

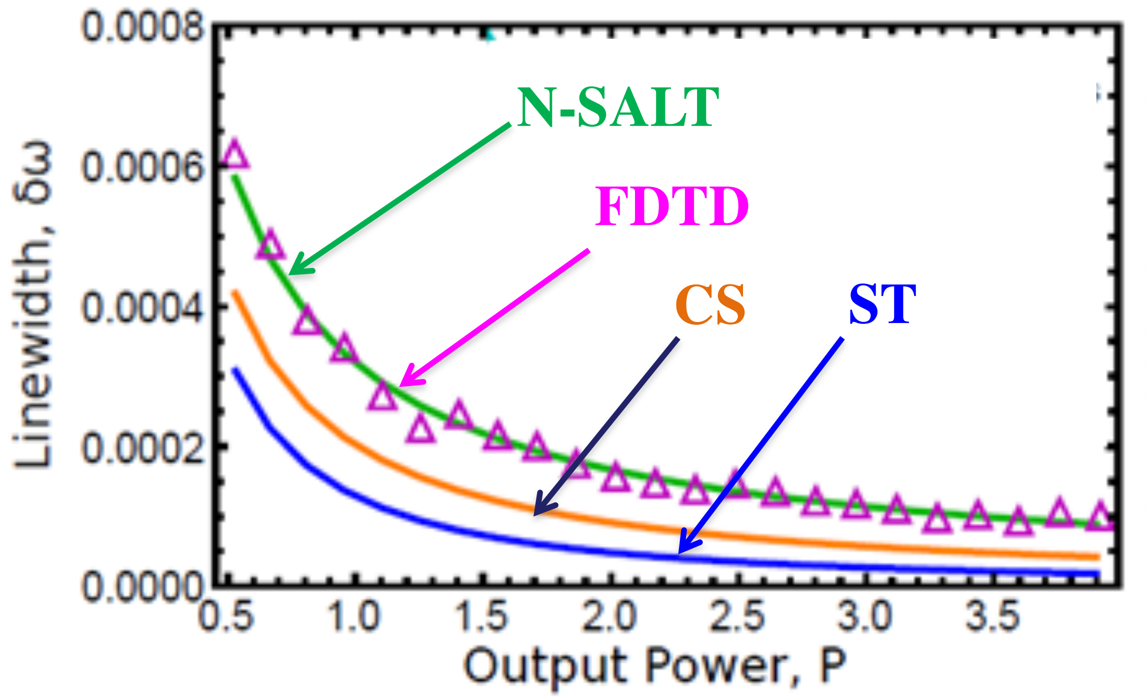
$$\mathrm{Im}\left[\frac{-i\omega_0^2\int\frac{\partial\varepsilon}{\partial|a|^2}\mathbf{E}_0^2}{\int\frac{\partial}{\partial\omega}(\omega^2\varepsilon)\mathbf{E}_0^2}\right] / \mathrm{Re}\left[\frac{-i\omega_0^2\int\frac{\partial\varepsilon}{\partial|a|^2}\mathbf{E}_0^2}{\int\frac{\partial}{\partial\omega}(\omega^2\varepsilon)\mathbf{E}_0^2}\right]$$

α factor

Brute-force validation

A. Cerjan et al., *Opt. Exp.* 23, 28316 (2015)

Brute-force simulations of Langevin–Maxwell–Bloch show excellent agreement with N-SALT linewidth formula



Only N-SALT captures **all relevant physics** in MB