18.336 Mid-term Exam

90 minutes. All problems have *equal weight*, so **don't spend too much time on one problem**.

Problem 1 (30 points): Velocity

Consider the leap-frog method for the (unsplit) scalar wave equation $u_{tt} = a^2 u_{xx}$:

$$\frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{\Delta t^2} = a^2 \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2}. \tag{1}$$

We showed in class that this is stable with $|g| = 1$ for $|a\lambda| < 1$, where $\lambda = \Delta t/\Delta x$.

(a) Find the group velocity $v_g = d\omega/d\beta$ for planewaves $e^{i(\beta x - \omega t)}$ and sketch v_g/a vs. $\beta\Delta x \in [0, \pi]$.

(b) Suppose we have an infinite computational cell and we start a wave via a Gaussian-pulse source $e^{-t^2}\sin(\omega t)$ for some ω, just like in pset 3. This sends out two pulses, one going left and one going right. After some time has elapsed, one of the pulses looks like the shape shown in figure 1. Is this pulse travelling to the right or to the left? Why? (Hint: the figure is to scale.)

Problem 2 (30 points): Stability

Calvin Q. Luss, a Harvard student, decides to add a PML absorbing region to the one-way wave equation $u_t = -au_x$, with periodic boundaries, by adding an absorption term $\sigma(x)$: $u_t = -au_x + \sigma u$.

(a) Explain why Cal got the sign wrong; i.e. show that for $\sigma > 0$ this leads to exponentially *growing* solutions.

(b) Regardless, show that the following finite-difference scheme for Cal's equation is stable (for constants $a \neq 0$ and $\sigma > 0$), and under what conditions:

$$\frac{u_m^{n+1} - u_m^n}{\Delta t} = -a\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2\Delta x} + \sigma u_m^n.$$

(c) "Wait a minute," Cal says. "If it's stable, by definition that means it doesn't blow up.

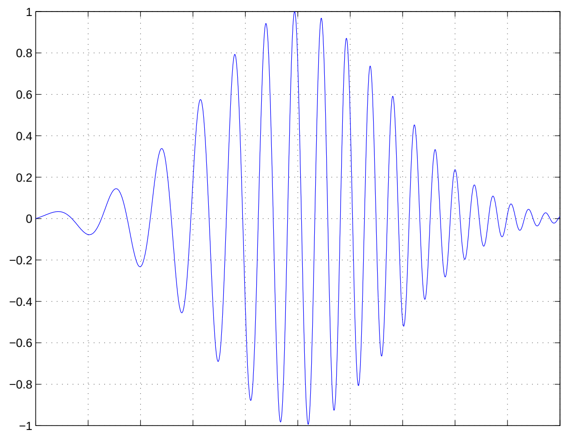

Figure 1: A Gaussian pulse that has propagated via Eq. (1) for some time. (For problem 1b.)

But Lax's theorem says a stable scheme has to converge to the exact solution, which you just claimed is exponentially growing—so either your (a) or your (b) answer must be wrong!" Is Cal correct? Why or why not?

Problem 3 (30 points): Accuracy

Suppose we are solving the Euler-Bernoulli beam equation $u_{xxxx} = f$ with periodic boundaries $u(x + 2\pi) = u(x)$ for the unknown displacement $u(x)$ and a given load $f(x)$. We'll use a spectral-collocation method where we express both u and f approximately by their DFT coefficients $\tilde{c}_k$ and $\tilde{d}_k$ for $k = -M, \ldots, M$:

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \approx u_M(x) = \sum_{k=-M}^{M} \tilde{c}_k e^{ikx}$$

$$f(x) = \sum_{k=-\infty}^{\infty} d_k e^{ikx} \approx f_M(x) = \sum_{k=-M}^{M} \tilde{d}_k e^{ikx}$$

These DFT coefficients are, of course, only approximations to the *exact* Fourier series coefficients c_k and d_k. Suppose that $f(x)$ is $\ell - 1$ times differentiable, so that $d_k = O(\frac{1}{|k|^\ell})$ as we showed in class. We also showed in class that $|d_k - \tilde{d}_k|$ is then $O(\frac{1}{M^\ell})$.

(a) Derive the rate at which the L_2 error

$$\|u - u_M\| = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} |u(x) - u_M(x)|^2 dx}$$

decreases with M. i.e. $\|u - u_M\| = O(\frac{1}{M^?})$?

Extra credit (5 points): Poisson

Suppose we're solving solving Poisson's equation $\nabla^2\phi = \rho$ with periodic boundary conditions $\phi(x) = \phi(x+2\pi)$ using some discretization of the ∇^2 operator (maybe center differences, or maybe some Galerkin approach), yielding an $N \times N$ linear equation $A\mathbf{c} = \mathbf{r}$ for the unknown coefficients $\mathbf{c}$.

(a) We implement this in Matlab and try to solve for $\mathbf{c}$ via `c = A \ r`. When we do this, however, Matlab prints the following scary message:
`Warning: Matrix is singular to working precision`
Why? (Hint: it has nothing really to do with the particular discretization.)

(b) How can we fix this, so that we are solving the same problem but are not solving singular equations?