Notes on the equivalence of norms

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If we are given two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ on some finite-dimensional vector space V over $\mathbb{C}$, a very useful fact is that they are always within a constant factor of one another. Specifically, there exists a pair of real numbers $0 < C_1 \le C_2$ such that, for all $x \in V$, the following inequality holds:

$$C_1\|x\|_b \le \|x\|_a \le C_2\|x\|_b.$$

Note that any *finite*-dimensional vector space, by definition, is spanned by a *basis* $e_1, e_2, \ldots, e_n$ where n is the *dimension* of the vector space. (The basis is often chosen to be orthonormal if we have an inner product, but non-orthonormal bases are fine too.) That is, any vector x can be written

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

where the α_i are some scalars depending on x.

Now, we can prove equivalence of norms in four steps, the last of which requires some knowledge of analysis. (I have seen other proofs as well, but they all require some theorem of analysis.)

Step 1: It is sufficient to consider $\|\cdot\|_b = \|\cdot\|_1$ (transitivity).

First, us define an L_1-style norm by

$$\|x\|_1 = \sum_{i=1}^{n} |\alpha_i|.$$

(It is easy to see this is a norm. The linear independence of any basis $\{e_i\}$ means that $x \neq 0 \iff \alpha_j \neq 0$ for some $j \iff \|x\|_1 > 0$. The triangle inequality and the scaling property are obvious and follow from the usual properties of L_1 norms on $\alpha \in \mathbb{C}^n$.)

We will show that it is sufficient for to prove that $\|\cdot\|_a$ is equivalent to $\|\cdot\|_1$, because norm equivalence is *transitive:* if two norms are equivalent to $\|\cdot\|_1$, then they are equivalent to each other. In particular, suppose both $\|\cdot\|_a$ and $\|\cdot\|_{a'}$ are equivalent to $\|\cdot\|_1$ for constants $0 < C_1 \le C_2$ and $0 < C_1' \le C_2'$, respectively:

$$C_1\|x\|_1 \le \|x\|_a \le C_2\|x\|_1,$$
$$C_1'\|x\|_1 \le \|x\|_{a'} \le C_2'\|x\|_1.$$

It immediately follows that

$$\frac{C_1'}{C_2}\|x\|_a \le \|x\|_{a'} \le \frac{C_2'}{C_1}\|x\|_a,$$

and hence $\|\cdot\|_a$ and $\|\cdot\|_{a'}$ are equivalent. Q.E.D.[1]

[1] While transitivity establishes upper/lower bounds for the relationship between $\|\cdot\|_a$ and $\|\cdot\|_{a'}$, and hence their equivalence, the constants $\frac{C_1'}{C_2}$ and $\frac{C_2'}{C_1}$ are not in general the tightest possible bounds even if the constants $C_{1,2}$ and $C_{1,2}'$ relating them to $\|\cdot\|_1$ were tight bounds. If you want to determine *tight* bounds relating $\|\cdot\|_a$ and $\|\cdot\|_{a'}$, you need to solve an optimization problem similar to Step 4 below for that particular pair of norms.

Step 2: It is sufficient to consider only x with $\|x\|_1 = 1$

We wish to show that

$$C_1\|x\|_1 \le \|x\|_a \le C_2\|x\|_1,$$

is true for *all* $x \in V$ for some C_1, C_2. It is trivially true for $x = 0$, so we need only consider $x \neq 0$, in which case we can divide by $\|x\|_1$ to obtain the condition

$$C_1 \le \|u\|_a \le C_2,$$

where $u = x/\|x\|_1$ has norm $\|u\|_1 = 1$. Q.E.D.

Step 3: Any norm $\|\cdot\|_a$ is continuous under $\|\cdot\|_1$

We wish to show that any norm $\|\cdot\|_a$ is a continuous function on V under the topology induced by the norm $\|\cdot\|_1$. That is, we wish to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x - x'\|_1 < \delta \implies |\|x\|_a - \|x'\|_a| < \epsilon.$$

We prove this in two steps. First, by the triangle inequality on $\|\cdot\|_a$, it follows that

$$\|x\|_a - \|x'\|_a = \|x' + (x - x')\|_a - \|x'\|_a \le \|x - x'\|_a$$

$$\|x'\|_a - \|x\|_a = \|x - (x - x')\|_a - \|x\|_a \le \|x - x'\|_a$$

and hence

$$|\|x\|_a - \|x'\|_a| \le \|x - x'\|_a$$

Second, applying the triangle inequality again, and writing $x = \sum_{i=1}^n \alpha_i e_i$ and $x' = \sum_{i=1}^n \alpha'_i e_i$ in our basis, we obtain

$$\|x - x'\|_a \le \sum_{i=1}^n |\alpha_i - \alpha'_i| \cdot \|e_i\|_a \le \|x - x'\|_1 \left(\max_i \|e_i\|_a\right).$$

Therefore, if we choose

$$\delta = \frac{\epsilon}{\max_i \|e_i\|_a},$$

it immediately follows that

$$\|x - x'\|_1 < \delta \implies |\|x\|_a - \|x'\|_a| \le \|x - x'\|_a < \epsilon.$$

Step 4: The maximum and minimum of $\|\cdot\|_a$ on the unit sphere

It is a standard theorem of analysis, the *extreme value theorem*, that a continuous function (e.g. $\|\cdot\|_a$, from step 3) on compact set (e.g. the unit "sphere" defined by $\{u \text{ for } \|u\|_1 = 1\}$, a closed and bounded set) must achieve a maximum and minimum value on the set (it cannot merely approach them). Let

$$C_1 = \min_{\|u\|_1=1} \|u\|_a,$$

$$C_2 = \max_{\|u\|_1=1} \|u\|_a.$$

Since $u \neq 0$ for $\|u\|_1 = 1$, it follows that $C_2 \ge C_1 > 0$ and

$$C_1 \le \|u\|_a \le C_2$$

as required by step 2. We are done!