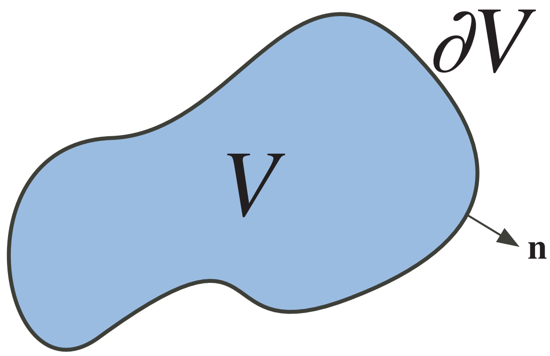

Figure 1: A volume V with a surface ∂V, and an outward unit normal vector $\mathbf{n}$ at each point on ∂V.

18.303 Problem Set 5

Due Monday, 27 October 2014.

Problem 1: Distributions

This problem concerns distributions as defined in the notes: continuous linear functionals $f\{\phi\}$ from test functions $\phi \in \mathcal{D}$, where $\mathcal{D}$ is the set of infinitely differentiable functions with compact support (i.e. $\phi = 0$ outside some region with finite diameter [differing for different ϕ], i.e. outside some finite interval $[a, b]$ in 1d).

(a) In this part, you will consider the function $f(x) = \begin{cases} \ln|x| & x \neq 0 \\ 0 & x = 0 \end{cases}$ and its (weak) derivative, which is connected to something called the Cauchy Principal Value.

(i) Show that $f(x)$ defines a regular distribution, by showing that $f(x)$ is locally integrable for all intervals $[a, b]$.

(ii) Consider the 18.01 derivative of $f(x)$, which gives $f'(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{undefined} & x = 0 \end{cases}$. Suppose we just set "$f'(0) = 0$" at the origin to define $g(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that this $g(x)$ is *not* locally integrable, and hence does not define a distribution.

But the weak derivative $f'\{\phi\}$ *must* exist, so this means that we have to do something different from the 18.01 derivative, and moreover $f'\{\phi\}$ is *not* a regular distribution. What is it?

(iii) Write $f\{\phi\} = \lim_{\epsilon \to 0^+} f_\epsilon\{\phi\}$ where $f_\epsilon\{\phi\} = \int_{-\infty}^{-\epsilon} \ln(-x)\phi(x)dx + \int_\epsilon^\infty \ln(x)\phi(x)dx$, since this limit exists and equals $f\{\phi\}$ for all ϕ from your proof in the previous part.[1] Compute the distributional derivative $f'\{\phi\} = \lim_{\epsilon \to 0^+} f'_\epsilon\{\phi\}$, and show that $f'\{\phi\}$ is precisely the *Cauchy Principal Value* (google the definition, e.g. on Wikipedia) of the integral of $g(x)\phi(x)$.

(iv) Alternatively, show that $f'\{\phi(x)\} = g\{\phi(x) - \phi(0)\} = \int_{-\infty}^{\infty} g(x)[\phi(x) - \phi(0)]dx$ (which *is* a well-defined integral for all $\phi \in \mathcal{D}$).

(b) In class, we only looked explicitly at 1d distributions, but a distribution in d dimensions $\mathbb{R}^d$ can obviously be defined similarly, as maps $f\{\phi\}$ from smooth localized functions $\phi(\mathbf{x})$ to numbers. Analogous to class, define the distributional gradient ∇f by $\nabla f\{\phi\} = f\{-\nabla\phi\}$.

Consider some finite volume V with a surface ∂V, and assume ∂V is differentiable so that at each point it has an outward-pointing unit normal vector $\mathbf{n}$, as shown in figure 1. Define a "surface delta function" $\delta(\partial V)\{\phi\} = \oint_{\partial V} \phi(\mathbf{x}) d^{d-1}\mathbf{x}$ to give the surface integral $\oint_{\partial V}$ of the test function.

[1] More explicitly, $f\{\phi\} - f_\epsilon\{\phi\} = \int_{-\epsilon}^{\epsilon} \ln|x|\phi(x)dx \leq (\max\phi)\int_{-\epsilon}^{\epsilon} \ln|x|dx \to 0$, since you should have done the something like the last integral explicitly in the previous part.

Suppose we have a regular distribution $f\{\phi\}$ defined by the function $f(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) & \mathbf{x} \in V \\ f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases}$, where we may have a discontinuity $f_2 - f_1 \neq 0$ at ∂V.

(i) Show that the distributional gradient of f is

$$\nabla f = \delta(\partial V)\left[f_1(\mathbf{x}) - f_2(\mathbf{x})\right]\mathbf{n}(\mathbf{x}) + \begin{cases} \nabla f_1(\mathbf{x}) & \mathbf{x} \in V \\ \nabla f_2(\mathbf{x}) & \mathbf{x} \notin V \end{cases},$$

where the second term is a regular distribution given by the ordinary 18.02 gradient of f_1 and f_2 (assumed to be differentiable), while the first term is the singular distribution

$$\delta(\partial V)\left[f_1(\mathbf{x}) - f_2(\mathbf{x})\right]\mathbf{n}(\mathbf{x})\{\phi\} = \oint_{\partial V}\left[f_1(\mathbf{x}) - f_2(\mathbf{x})\right]\mathbf{n}(\mathbf{x})\phi(\mathbf{x})d^{d-1}\mathbf{x}.$$

You can use the integral identity that $\int_V \nabla\psi d^d\mathbf{x} = \oint_{\partial V} \psi \mathbf{n} d^{d-1}\mathbf{x}$ to help you integrate by parts.

(ii) Defining $\nabla^2 f\{\phi\} = f\{\nabla^2\phi\}$, derive a similar expression to the above for $\nabla^2 f$. Note that you should have one term from the discontinuity $f_1 - f_2$, and another term from the discontinuity $\nabla f_1 - \nabla f_2$. (Recall how we integrated ∇^2 by parts in class some time ago.)

Problem 2: Green's functions

Consider Green's functions of the self-adjoint indefinite operator $\hat{A} = -\nabla^2 - \omega^2$ ($\kappa > 0$) over all space ($\Omega = \mathbb{R}^3$ in 3d), with solutions that $\to 0$ at infinity. (This is the multidimensional version of problem 2 from pset 5.) As in class, thanks to the translational and rotational invariance of this problem, we can find $G(\mathbf{x}, \mathbf{x}') = g(|\mathbf{x} - \mathbf{x}'|)$ for some $g(r)$ in spherical coordinates.

(a) Solve for $g(r)$ in 3d, similar to the procedure in class.

(i) Similar to the case of $\hat{A} = -\nabla^2$ in class, first solve for $g(r)$ for $r > 0$, and write $g(r) = \lim_{\epsilon \to 0^+} f_\epsilon(r)$ where $f_\epsilon(r) = 0$ for $r \leq \epsilon$. [Hint: although Wikipedia writes the spherical $\nabla^2 g(r)$ as $\frac{1}{r^2}(r^2 g')'$, it may be more convenient to write it equivalently as $\nabla^2 g = \frac{1}{r}(rg)''$, as in class, and to solve for $h(r) = rg(r)$ first. Hint: if you get sines and cosines from this differential equation, it will probably be easier to use complex exponentials, e.g. $e^{i\omega r}$, instead.]

(ii) In the previous part, you should find two solutions, both of which go to zero at infinity. To choose between them, remember that this operator arose from a $e^{-i\omega t}$ time dependence. Plug in this time dependence and impose an "outgoing wave" boundary condition (also called a Sommerfield or radiation boundary condition): require that waves be traveling *outward* far away, not *inward*.

(iii) Then, evaluate $\hat{A}g = \delta(\mathbf{x})$ in the distributional sense: $(\hat{A}g)\{q\} = g\{\hat{A}q\} = q(0)$ for an arbitrary (smooth, localized) test function $q(\mathbf{x})$ to solve for the unknown constants in $g(r)$. [Hint: when evaluating $g\{\hat{A}q\}$, you may need to integrate by parts on the radial-derivative term of $\nabla^2 q$; don't forget the boundary term(s).]

(b) Check that the $\omega \to 0^+$ limit gives the answer from class.