

# 18.303: Notes on the 1d-Laplacian Green's function

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In class, we solved for the Green's function  $G(x, x')$  of the 1d Poisson equation  $-\frac{d^2}{dx^2}u = f$  where  $u(x)$  is a function on  $[0, L]$  with Dirichlet boundaries  $u(0) = u(L) = 0$ . We obtained:

$$G(x, x') = \begin{cases} -\frac{x'}{L}(x-L) & x > x' \\ \left(1 - \frac{x'}{L}\right)x & x \leq x' \end{cases} = \begin{cases} x' \left(1 - \frac{x}{L}\right) & x > x' \\ x \left(1 - \frac{x'}{L}\right) & x \leq x' \end{cases},$$

which can be derived in any number of ways. Once we understand delta “functions” and distributions, we can derive it from  $-\frac{\partial^2}{\partial x^2}G(x, x') = \delta(x - x')$ , where the  $\delta(x - x')$  yields a slope discontinuity of 1 in  $G$  at  $x = x'$ . Instead, we first derived it the “long” way, by taking a width- $\Delta x$  approximation for the delta function ( $= 1/\Delta x$  in  $[x', x' + \Delta x]$  and  $= 0$  otherwise) on the right-hand side, solving, and then taking the limit  $\Delta x \rightarrow 0$  on the *solution* (whereas the limit  $\Delta x \rightarrow 0$  of the right-hand side does not exist as an ordinary function).

Regardless of how it is obtained, we get the  $G(x, x')$  above. Clearly, this  $G$  satisfies the reciprocity property  $G(x, x') = G(x', x)$ , which (as shown in class) follows from the fact that the Laplacian is real-symmetric with these boundary conditions. Physically,  $G(x, x')$  could be thought of as a “plucked string:” a stretched string where you have concentrated a finite force at one “point”  $x'$  in the string.

It is also good to check that

$$u(x) = \int_0^L G(x, x') f(x') dx'$$

satisfies  $-\frac{d^2}{dx^2}u = f$  for any  $f(x)$ . Note that  $u(0) = u(L) = 0$  automatically, since  $G(0, x') = G(L, x') = 0$  by construction. If we use delta functions, taking the second derivative of  $G$  is easy. However, let's do it the “hard way” instead, performing one derivative at a time and avoiding differentiating any discontinuous function. We can do this by breaking the integral up into two pieces  $[0, x]$  and  $[x, L]$  based on the two pieces of  $G(x, x')$ :

$$u(x) = \left(1 - \frac{x}{L}\right) \int_0^x x' f(x') dx' + x \int_x^L \left(1 - \frac{x'}{L}\right) f(x') dx',$$

where we have pulled the  $x$  terms out of the  $x'$  integrals. The first derivative is then:

$$u'(x) = -\frac{1}{L} \int_0^x x' f(x') dx' + \int_x^L \left(1 - \frac{x'}{L}\right) f(x') dx' + \cancel{\left(1 - \frac{x}{L}\right) x f(x)} - \cancel{x \left(1 - \frac{x}{L}\right) f(x)},$$

where the terms on the second line [which cancel each other] come from the derivatives of the integration limits. [Recall the fundamental theorem of calculus:  $\frac{d}{dx} \int^x h(x') dx' = h(x)$  and  $\frac{d}{dx} \int_x h(x') dx' = -h(x)$  for any  $h$ .] Since  $x$  only appears in the integration limits in  $u'$ , the second derivative is then given by the fundamental theorem of calculus again:

$$u''(x) = -\cancel{\frac{1}{L} x f(x)} - \left(1 - \frac{x}{L}\right) f(x) = -f(x),$$

and thus  $-u'' = f$  as desired.

More abstractly, the key properties are that  $G(x, x')$  is continuous:  $G(x, x^-) = G(x, x^+)$  [where  $x^\pm$  denote the limits as  $x' \rightarrow x$  from the left and right], while  $G'(x, x') = \frac{\partial}{\partial x} G(x, x')$  has a jump discontinuity  $G'(x, x^-) = G'(x, x^+) - 1$ , and  $G''(x, x') = \frac{\partial^2}{\partial x^2} G(x, x') = 0$  for  $x \neq x'$ . That means:

$$u(x) = \int_0^x G(x, x') f(x') dx' + \int_x^L G(x, x') f(x') dx',$$

$$u'(x) = \int_0^x G'(x, x') f(x') dx' + \int_x^L G'(x, x') f(x') dx' + \cancel{G(x, x^-) f(x)} - \cancel{G(x, x^+) f(x)},$$

$$u''(x) = G'(x, x^-) f(x) - G'(x, x^+) f(x) = -f(x)$$

as desired (omitting the  $G'' = 0$  terms from the last line).

So, all of this can be solved without delta functions or discontinuities, and people did this historically. But it is *much easier* if we don't have to continually dodge discontinuities and can differentiate them naturally. Generalizing functions as distributions (or solving the "weak form" of the PDE), and setting up the rules for differentiating distributions, will allow us to do that.