EQUATIONS OF GENUS 4 CURVES FROM THEIR THETA CONSTANTS

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Abstract. In this article we give explicit formulas for the equations of a generic genus 4 curve in terms of its theta constants. The method uses 20 tritangent planes as well as the Prym construction and the beautiful classical geometry around it.

Contents

1. Introduction

Recall that over $k = \mathbb{C}$ the theta constants of a principally polarized abelian variety (p.p.a.v.) of dimension g with (small) period matrix $\tau \in \mathfrak{H}_g$ are given by the $2^{g-1}(2^g + 1)$ numbers $\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](0, \tau)$ with $a, b \in \frac{1}{2}\mathbb{Z}^g$ running through a set of representatives of pairs in $\frac{1}{2}\mathbb{Z}^g\,/\,\mathbb{Z}^g$ satisfying $4a^tb \equiv 0 \bmod 2$. Here $\vartheta\big[\begin{smallmatrix}a\\b\end{smallmatrix}\big]$ is the Riemann theta function with characteristics, given by

$$\vartheta\big[\begin{smallmatrix}a\\b\end{smallmatrix}\big](z,\tau) = \sum_{n\in\mathbb{Z}^g} \exp\left(\pi i(n+a)^t\tau(n+a) + 2\pi i(n+a)^t(z+b)\right)\,.$$

Mumford [Mum66] gave a purely algebraic definition of the theta constants that works over any algebraically closed field k of characteristic $\neq 2$. He also showed that the theta constants determine the principally polarized abelian variety uniquely.

The goal of this article is to provide closed formulas for recovering explicit equations of a generic genus 4 curve C, given the values of its algebraic theta constants. The result is a quadric Q and a cubic Γ in $\mathbb{P}^3$ such that $Q \cap \Gamma$ is equal to the image of the canonical embedding of C.

The problem of recovering the equations of a curve from its theta constants was classically studied by Rosenhain (genus 2) and Aronhold-Weber (plane quartics). Later Takase [Tak96] generalized Rosenhain's work to arbitrary hyperelliptic curves. In all these cases they exploit the fact that the moduli space of these curves equipped with a full level 2 structure is rational. In the hyperelliptic case this rationality follows from the existence of Weierstrass equations, and for plane quartics the formulas of Aronhold give an explicit rational parametrization.

In genus 4 we are faced with a different situation, as the moduli space of genus 4 curves with a full level 2 structure is not unirational. (This will be proved in a later article [HPS24a].) This fact may explain why the general problem has remained open, only garnering results in special cases. For instance, Schottky [Sch88] managed to solve the problem in the case where one of the theta constants vanishes, which is equivalent to the canonical quadric being a cone.

Another phenomenon not appearing in the theory of plane quartics is that not every p.p.a.v. is a Jacobian when $g \geqslant 4$. In fact, a 4-dimensional (indecomposable) p.p.a.v. is the Jacobian of a smooth curve if and only if the Schottky modular form vanishes [FR70, Chapter 5][Igu81][Fre83].

Recall from [FR70, Chapter 5] that the Schottky modular form is related to the Prym construction as follows. Given an étale double cover $\tilde{C} \to C$, the Prym variety $\mathrm{Prym}(\tilde{C}/C) = \ker(\mathrm{Nm} : \mathrm{Jac}(\tilde{C}) \to \mathrm{Jac}(C))^\circ$ is in a natural way a principally polarized abelian variety. In particular, the theta constants of $\mathrm{Prym}(\tilde{C}/C)$ must satisfy the quartic Riemann identities. The Schottky modular form is derived from these identities by expressing the theta constants of $\mathrm{Prym}(\tilde{C}/C)$ in terms of the theta constants of $\mathrm{Jac}(C)$.

This suggests that the Prym construction could play a role in the problem of reconstructing the curve from the theta constants, and this is the approach we pursue here. Furthermore, we will use a beautiful classical geometric construction concerning the Prym due to Caporali, Wirtinger, P. Roth (see [Cob29] for a classical textbook reference), and W. P. Milne [Mil23]. The modern development was pioneered by Recillas [Rec71] and henceforth it is commonly referred to as Recillas's trigonal construction. In this article we mainly follow the articles of Catanese [Cat81] and Bruin–Sertöz [BS20] that are closer to the classical works.

1.1. **Motivation and applications.** In addition to the intrinsic value of recovering a genus 4 curve from its theta constants (or its period matrix), our method also has several applications. It is interesting in arithmetic and complex geometry to be able find higher genus curves with special properties. Being able to reconstruct a curve from the theta constants (or period matrix) of its Jacobian allows us to construct Jacobians with interesting properties and find corresponding curves that inherit those properties. Below we list a couple of examples that illustrate this strategy. These examples were computed using our Magma [BCP97] implementation of our algorithm [HPS24c].

- **Gluing**. In [HSS21], the first and third authors, together with Sijsling, described several methods for gluing genus 1 and genus 2 curves along their torsion. More precisely, let X_1 be a curve of genus 1 and let X_2 be a curve of genus 2 with Jacobian varieties J_1 and J_2, respectively. Gluing X_1 and X_2 along their 2-torsion means finding a curve X_3 with Jacobian J_3 and an isogeny $\phi : J_1 \times J_2 \to J_3$ such that $\ker \phi$ is contained in the 2-torsion of $J_1 \times J_2$.

  The analytic method described in [HSS21, §2] for computing the period matrices of such gluings easily generalizes to higher genera. However, after constructing the relevant period matrix, it still remains to (a) determine whether the corresponding abelian variety is a Jacobian; and (b) if so, recover the equation of the corresponding curve. In Example 4.1, we show how our algorithm can be used to accomplish this last step, allowing one to glue two genus 2 curves along their 2-torsion.
- **Constructing modular abelian varieties**. Shimura associates to a Hecke newform f with level $\Gamma_1(N)$ a subvariety A_f of the modular Jacobian $J_1(N)$. This A_f is simple and $\mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$ contains the field K generated by the coefficients of the q-expansion of f. Furthermore, K is totally real and one has $[K : \mathbb{Q}] = \dim(A_f)$. Conversely, Khare and Wintenberger [KW09a, Corollary 10.2], [KW09b] have shown that every simple abelian variety A over $\mathbb{Q}$ with the property that $\mathrm{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q}$ contains an RM field K such that $[K : \mathbb{Q}] = \dim(A)$, must be modular. This means that there exists a Hecke newform f with level $\Gamma_1(N)$ such that A is isogenous to A_f. Furthermore, N is then equal to the conductor of A.

  John Cremona [Cre97] gave a method for computing the period matrix of such modular abelian varieties and used this to construct many examples of modular elliptic curves. Since the computation of the period matrix works for general g, one can use the method of the present article to compute curves of genus 4 whose Jacobian is isomorphic to A_f. Although there is no general reason why A_f should be a Jacobian, Noam Elkies (private communication) provided an example of an eigenform f such that A_f is the Jacobian of a genus 4 curve. In Example 4.2 we recover this genus 4 curve over $\mathbb{Q}$ whose Jacobian is a modular abelian variety with real multiplication by the maximal totally real subfield of $\mathbb{Q}(\zeta_{15})$.

1.2. **Previous work.** Lehavi [Leh10] gave an effective method for reconstructing a non-hyperelliptic genus 4 curve from its tritangent planes, which partially inspired this paper. Lehavi's method is based on the cartesian diagrams

$$\begin{tikzcd} S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\times_{\mathrm{H}^0(\Omega_C^{\otimes 2})} S^2\,\mathrm{H}^0(\Omega_C) \arrow[r, "\varphi"] \arrow[d, hook] & S^2\,\mathrm{H}^0(\Omega_C\otimes\eta) \arrow[d, hook] \\ S^2\,\mathrm{H}^0(\Omega_C) \arrow[r] & \mathrm{H}^0(\Omega_C^{\otimes 2}) \end{tikzcd} \tag{1.1}$$

where η runs through all the 255 non-trivial two-torsion points in $\mathrm{Jac}(C)[2]\setminus\{0\}$. He focuses on the vector space

$$S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\times_{\mathrm{H}^0(\Omega_C^{\otimes 2})} S^2\,\mathrm{H}^0(\Omega_C)$$

rather than the map φ. The novelty of the present paper is a link between the map φ and the Prym construction. This allows us prove that the knowledge of φ for just one two-torsion point η is sufficient to recover the curve C.

Various mathematicians have written articles about reconstructing genus 4 curves using theta functions (starting from the period matrices of their Jacobians). Agostini, Çelik, and Eken [AÇE22] describe a method using Dubrovin threefolds to numerically reconstruct a curve of arbitrary genus from the period matrix their Jacobian. They use theta derivatives up to order 4 to compute degree 4 polynomials vanishing on the curve. In genus 4 they require Gröbner basis computations to write the equations as the intersection of a quadric and a cubic.

In [Kem86] Kempf describes a method to reconstruct a genus 4 curve using the singular point of the theta divisor. This was implemented in the work of Chua, Kummer, and Sturmfels [CKS18]. In order to find this singular point they need to solve a highly non-linear system for computing the point where both a theta function and all its derivatives vanish.

Our method is different in that we give explicit formulas for the equations of the reconstructed curve C in terms of its theta constants. In order to do this we use only algebraic operations, namely elementary arithmetic, extraction of square roots, and solving linear systems of equations. We directly obtain a model for C as the intersection of a quadric and cubic in $\mathbb{P}^3$ without the need for further computation.

There are several advantages of using just theta constants and no derivatives. First, it is much better understood how to compute theta constants efficiently; for example, Labrande and Thomé's algorithm [LT16]. In a forthcoming article, Elkies and Kieffer [EK24] will describe an algorithm for computing theta constants with quasi-linear complexity. Second, when using derivatives up to order n, the total number of evaluations is multiplied by $\binom{g+n}{n}$; the number of possibilities of taking a partial derivative.

1.3. **Structure of the article.** The article is organized as follows. In Section 2 we begin by defining the objects that will play key roles in our construction. In Section 3 we describe our main result: an algorithm that takes theta constants of the Jacobian of a genus 4 curve as its input and, using only elementary arithmetic, extraction of square roots, and solving linear systems of equations, outputs the corresponding quadric and cubic whose solution set is equal to the image of the canonical embedding of the corresponding curve. In Section 4 we present two examples as an application of our method. First, we construct a gluing with an interesting endomorphism algebra, and then the curve whose Jacobian is the modular abelian variety discussed above, i.e., with real multiplication by the maximal real subfield of $\mathbb{Q}(\zeta_{15})$. Finally, in Section 5 we briefly discuss how our method can be applied to find more interesting curves, as well as how it can be generalized to higher genus.

For an outline of the various steps in our method, see Algorithm 3.1.

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2. Preliminaries

2.1. **The Prym canonical map.** This section summarizes results from Catanese's article [Cat81]. Let C be a non-hyperelliptic genus 4 curve, $\eta \in \mathrm{Jac}(C)[2] \setminus \{0\}$ be a non-trivial 2-torsion point. The Prym canonical map is the map

$$\phi_\eta : C \longrightarrow \mathbb{P}(\mathrm{H}^0(C, \Omega_C \otimes \eta)) \cong \mathbb{P}^2$$

associated to the linear system $|\Omega_C \otimes \eta|$. The map ϕ_η is either a ramified degree 2 map onto a smooth cubic (the bielliptic case) or a birational map onto a singular plane sextic. (For details, see [Cat81, §1].) We will discuss the bielliptic case in Section 2.4 below. Thus assume now we are in the non-bielliptic case and hence the Prym canonical curve $\mathrm{im}(\phi_\eta)$ is a singular plane sextic embedded in $\mathbb{P}^2$. Let $b_1 : S \to \mathbb{P}^2$ be the repeated blow-up of $\mathbb{P}^2$ in the singular points of the Prym canonical curve. Denote by $E_1, \ldots, E_k$ the total transform of the exceptional divisors of the successive blow-ups.

On S we can consider the line bundle $\mathcal{L} = b_1^*\mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}(-\sum_{j=1}^{k}(r_j - 1)E_j)$ where r_j is the multiplicity of the singular point blown up in the jth step. The line bundle $\mathcal{L}$ induces a map $S \to \mathbb{P}^3$ whose image, which we denote by Γ_η, is a cubic symmetroid [Cat81, p. 37]. We recall the definition of a cubic symmetroid.

Definition 2.1. A *cubic symmetroid* is a cubic surface V which is the vanishing scheme of the determinant of a symmetric 3×3 matrix of linear forms. That is, there exists a symmetric 3×3 matrix A with entries $a_{ij} \in k[x_0, \dots, x_3]$ homogeneous of degree 1 such that $V = \mathcal{V}(\det(A))$.

We define a rational map $c : \mathbb{P}^2 \dashrightarrow \Gamma_\eta$ as the composition of b_1^{-1} with the map $b_2 : S \to \Gamma_\eta$. The situation is summarized by the following commutative diagram.

$$\begin{tikzcd} & S \arrow[dl, "b_1"'] \arrow[dr, "b_2"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "c"] & & \Gamma_\eta \subseteq \mathbb{P}^3 \end{tikzcd} \tag{2.1}$$

The map $b_2 : S \to \Gamma_\eta$ is a blow-up; this follows from the observations in [Cat81, p. 37].

We will describe the situation for a generic C. In this case, the Prym canonical image $\operatorname{im}(\phi_\eta)$ has six nodes that are the pairwise intersection of four lines, say $n_1, \dots, n_4$. The surface S is the blow-up in these six nodes. The strict transforms of the n_i under b_1 are (-2)-curves. These are contracted by b_2 to A_1 singularities on Γ_η.

There are two possible degenerate cases. First, it could happen that two of the lines n_i coincide. In this case, the two corresponding A_1-singularities combine to form a single A_3-singularity. Then Γ_η has three singular points: Two of type A_1 and one of type A_3.

Second, three of the lines n_i could coincide. In that case, Γ_η has one singularity of type A_1 and one of type A_5. For more details, see [Cat81, p. 33].

The main consequence of Catanese's construction is the following theorem that he attributes to Wirtinger–Coble–Recillas.

Theorem 2.2. *The map $\eta \mapsto \Gamma_\eta$ gives a bijection between:*

- $\operatorname{Jac}(C)[2] \setminus \{0\}$.
- *Irreducible cubic symmetroids containing C.*

Proof. See [Cat81, Theorem 1.5]. □

2.2. **Two linear maps.** We will now define a linear map φ that maps certain quadratic forms on the right $\mathbb{P}^3$ to quadratic forms on the left $\mathbb{P}^2$.

We begin by considering the following cartesian diagram.

$$\begin{tikzcd} S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\times_{\mathrm{H}^0(\Omega_C^{\otimes 2})} S^2\,\mathrm{H}^0(\Omega_C) \arrow[r] \arrow[d, hook] & S^2\,\mathrm{H}^0(\Omega_C\otimes\eta) \arrow[d, hook] \\ S^2\,\mathrm{H}^0(\Omega_C) \arrow[r] & \mathrm{H}^0(\Omega_C^{\otimes 2}) \end{tikzcd} \tag{2.2}$$

where the maps $S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\to\mathrm{H}^0(C,\Omega_C^{\otimes 2})$, $S^2\,\mathrm{H}^0(\Omega_C)\to\mathrm{H}^0(C,\Omega_C^{\otimes 2})$ are given by multiplication. The injectivity of the right vertical arrow expresses the fact that there are no quadrics vanishing on $\mathrm{im}(\phi_\eta)$.

Definition 2.3. Following Lehavi [Leh10, p. 2] we define $V_{C,\eta}\subset S^2\,\mathrm{H}^0(C,\Omega_C)$ as the image of the embedding

$$S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\times_{\mathrm{H}^0(\Omega_C^{\otimes 2})} S^2\,\mathrm{H}^0(\Omega_C)\hookrightarrow S^2\,\mathrm{H}^0(C,\Omega_C).$$

We define Lehavi's map φ to be the projection map

$$\varphi: V_{C,\eta}\to S^2\,\mathrm{H}^0(\Omega_C\otimes\eta)\,.$$

From the facts that $\dim(S^2\,\mathrm{H}^0(C,\Omega_C))=10$, $\dim(S^2\,\mathrm{H}^0(C,\Omega_C\otimes\eta))=6$ and $\dim(\mathrm{H}^0(C,\Omega_C^{\otimes 2}))=9$ one can readily compute that $\dim(V_{C,\eta})=7$. Furthermore, it will later turn out to be useful that $\ker(\varphi)$ is one-dimensional and generated by the quadratic form Q vanishing on C.

Next, we define a right inverse ψ for φ using the following lemma.

Lemma 2.4. *Assume that Γ_η is generic. Let us denote by $N_i\subset S$, $i=1,\dots,4$ the exceptional divisors of b_2. Then $b_1^*\mathcal{O}(2)\cong b_2^*\mathcal{O}(2)\otimes\mathcal{O}_S(-\sum N_i)$. If Γ_η is not generic, then the N_i have to be taken with appropriate multiplicities.*

Proof. Consider the curves $n_i=b_1(N_i)$. Then, by the discussion following Definition 2.1, the $n_i\in\mathbb{P}^2$ are lines such that the 6 singular points of $\mathrm{im}(\phi_\eta)$ are the intersection points of pairs from $\{n_i\mid i=1,\dots,4\}$.

By definition of b_2 we have an isomorphism

$$b_2^*\mathcal{O}(1)\cong\mathcal{L}=b_1^*\mathcal{O}_{\mathbb{P}^2}(3)\otimes\mathcal{O}_S\left(-\sum_{j=1}^{6}E_j\right)\,.$$

These two facts imply the lemma. $\square$

We define $W_\eta\subset S^2\,\mathrm{H}^0(C,\Omega_C)$ to be the vector space of quadrics vanishing in the nodes of Γ_η. Define the map $\psi: S^2\,\mathrm{H}^0(C,\Omega_C\otimes\eta)\longrightarrow W_\eta$ to be the composition

$$\mathrm{H}^0(\mathbb{P}^2,\mathcal{O}(2))\xrightarrow{b_1^*}\mathrm{H}^0(S,b_1^*\mathcal{O}(2))\cong\mathrm{H}^0\left(S,b_2^*\mathcal{O}(2)\otimes\mathcal{O}_S(-\sum N_i)\right)=:W_\eta\,.$$

Geometrically this means the following: for any $q\in S^2\,\mathrm{H}^0(C,\Omega_C\otimes\eta)$, the image $\psi(q)$ is the unique quadratic form that cuts out the curve

$$c(\mathcal{V}(q))=b_2(b_1^{-1}(\mathcal{V}(q)))\subset\Gamma_\eta\,.$$

Remark 2.5. The map ψ is only well-defined up to multiplication by a scalar because of the arbitrary choice of an isomorphism $b_1^*\mathcal{O}(2) \cong b_2^*\mathcal{O}(2) \otimes \mathcal{O}_S(-\sum N_i)$. We resolve this ambiguity in the proof of the following lemma.

The following lemma proves that ψ is a right-inverse of φ.

Lemma 2.6. *There exists a natural way of removing the ambiguity by a scalar in the definition of ψ. Furthermore, with this choice one has*

$$\varphi \circ \psi = \mathrm{id}\ .$$

Proof. We start by considering the diagram

$$\begin{tikzcd} & C \arrow[dl, "\phi_\eta"'] \arrow[dr] & \\ \mathbb{P}^2 \arrow[rr, dashed, "c"] & & \mathbb{P}^3 \end{tikzcd}$$

where the left map is the Prym canonical embedding. This diagram is commutative as a consequence of [Cat81, Equation 1.13]. Therefore the diagram

$$\begin{tikzcd} & C \arrow[ddl, "\phi_\eta"'] \arrow[d, hook] \arrow[ddr] & \\ & S \arrow[dl, "b_1"] \arrow[dr, "b_2"'] & \\ \mathbb{P}^2 \arrow[rr, dashed, "c"] & & \mathbb{P}^3 \end{tikzcd} \tag{2.3}$$

commutes. Now recall from Lemma 2.4 that the definition of ψ is based on the isomorphism

$$b_1^*\mathcal{O}(2) \cong b_2^*\mathcal{O}(2) \otimes \mathcal{O}_S(-\sum N_i)$$

which has the ambiguity by a scalar. Restricting this isomorphism to C gives an isomorphism

$$(\Omega_C \otimes \eta)^2 \cong \Omega_C^{\otimes 2} \cong \Omega_C^{\otimes 2}\ .$$

(Notice that C cannot go through the nodes of Γ_η because then $Q \cap \Gamma_\eta$ would be singular.) We can now remove the ambiguity in the definition of ψ by requiring that the latter isomorphism be the identity. This, together with the commutativity of (2.3), implies that the diagram

$$\begin{tikzcd} & S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta) \arrow[dl, "\psi"'] \arrow[d] \\ S^2\,\mathrm{H}^0(C, \Omega_C) \arrow[r] & \mathrm{H}^0(C, \Omega_C^{\otimes 2}) \end{tikzcd}$$

commutes.

Consider now the diagram defining φ.

$$\begin{tikzcd} V_{C,\eta} \arrow[r, "\varphi"] \arrow[d, hook] & S^2\,\mathrm{H}^0(\Omega_C \otimes \eta) \arrow[d, hook] \\ S^2\,\mathrm{H}^0(\Omega_C) \arrow[r] & \mathrm{H}^0(\Omega_C^{\otimes 2}) \end{tikzcd}$$

Since this diagram also commutes and the map $S^2\,\mathrm{H}^0(C,\Omega_C\otimes\eta) \to \mathrm{H}^0(C,\Omega_C^{\otimes 2})$ is injective, we conclude that

$$\varphi \circ \psi = \mathrm{id}\,,$$

which proves the lemma. □

The situation is summarized by the following diagram.

$$\begin{tikzcd} V_{C,\eta} \arrow[r, shift left, "\varphi"] \arrow[d, hook] & S^2\,\mathrm{H}^0(\Omega_C \otimes \eta) \arrow[l, shift left, "\psi"] \arrow[d, hook] \\ S^2\,\mathrm{H}^0(\Omega_C) \arrow[r] & \mathrm{H}^0(\Omega_C^{\otimes 2}) \end{tikzcd}$$

2.3. **Milne's bijection.** Let $\widetilde{C} \to C$ be the étale double cover associated to $\eta \in \mathrm{Jac}(C)[2]\setminus\{0\}$, so $\widetilde{C}$ has genus 7. Since the Prym variety $\mathrm{Prym}(\widetilde{C}/C)$ is principally polarized, then, up to quadratic twist, $\mathrm{Prym}(\widetilde{C}/C)$ is isomorphic to the Jacobian of some genus 3 curve X (see [OU73] and [BR11]). (The Schottky-Jung relations imply that the Prym is indecomposable.) Since we have an isomorphism $\mathrm{H}^0(X,\Omega_X) \cong \mathrm{H}^0(C,\Omega_C\otimes\eta)$, the curve X canonically maps to $\mathbb{P}(\mathrm{H}^0(C,\Omega_C\otimes\eta)) \cong \mathbb{P}^2$.

Remark 2.7. The Recillas trigonal construction (see [Cob29, §50] or [BS20, Lemma 5.6]) gives an explicit geometric construction of X as a plane quartic (generic case), but this is not used in the present article.

In order to state the main theorem of this section we need the following definition. Given a tritangent plane H of C, then $H.C = 2D$ where D is a divisor of degree 3. We say that a pair of tritangents H, H' *differ by* $\eta \in \mathrm{Jac}(C)[2]$ if $D - D'$ is linearly equivalent to η, where $H.C = 2D$ and $H'.C = 2D'$.

Theorem 2.8. *If X is a plane quartic, there is a bijection*

$$\left\{\ \textit{Pairs of tritangent planes of } C \textit{ differing by } \eta\ \right\} \xrightarrow{\sim} \left\{\ \textit{Bitangents of } X\ \right\}$$

such that for any pair H, H' viewed as elements of $\mathrm{H}^0(C,\Omega_C)$ mapping to a bitangent $\ell \subset \mathbb{P}(\mathrm{H}^0(C,\Omega_C\otimes\eta))$ viewed as an element of $\mathrm{H}^0(C,\Omega_C\otimes\eta)$, we have:

(i) *The product $HH' \in S^2\,\mathrm{H}^0(C,\Omega_C)$ lies in $V_{C,\eta}$.*
(ii) *There exists $\lambda \in k^\times$ such that*

$$\varphi(HH') = \lambda\ell^2. \tag{2.4}$$

Proof. The bijection between pairs of tritangent planes differing by η and bitangents of X was discovered by W. P. Milne [Mil23] (see [BS20, Theorem 7.1] for a modern proof). The assertion (i) is due to Lehavi [Leh10, p. 2]. To prove (ii) one can use the argument from [BS20, Theorem 7.1], the geometric interpretation of the map ψ, and Lemma 2.6. □

Remark 2.9. A similar statement holds when X is a hyperelliptic curve. The main difference is that one has to replace the bitangents of X with the lines through the pairs of Weierstrass points of X in the bijection (see [BS20, Section 7.2]).

Milne's theorem will play a key role in reconstructing the genus 4 curve because it links the map φ to the Prym. Since the bitangent lines of X can be computed with classical formulas, namely the Schottky-Jung relations [FR70, Theorem 1], and the Aronhold-Weber formulas [Dol12, Theorem 6.1.9] [Fio16, Theorem 2], the information on the righthand side of the identity

$$\varphi(HH') = \lambda \ell^2$$

can be readily computed, except for the unknown constant λ. A central step of our method is to compute the map φ by interpolating this identity through ten pairs of tritangent planes and their corresponding bitangents.

Remark 2.10. The formulas in [Dol12, Theorem 6.1.9, Equations (1) - (7)] contain typos that will be corrected in a future edition. We thank Igor Dolgachev for providing us with these corrections.

2.4. **The bielliptic case.** We will explain now the modifications that have to be made in the bielliptic case. All the facts in this section are due to Catanese and Bruin–Sertöz; see [Cat81, (1.9)], [BS20, §4.2] for proofs. Let C be a smooth non-hyperelliptic genus 4 curve and $\eta \in \mathrm{Jac}(C)[2] \setminus \{0\}$. We will say that the pair (C, η) is *bielliptic* if there is a degree two map $\pi : C \to E$ onto a smooth genus 1 curve and a two-torsion point $\eta_0 \in \mathrm{Jac}(E)$ such that $\pi^*\eta_0 = \eta$.

Then (C, η) is bielliptic if and only if the Prym canonical map $\phi_\eta : C \to \mathbb{P}^2$ factors through a degree two map onto a smooth plane cubic $E \subset \mathbb{P}^2$. In this situation, one can still obtain Γ_η as the cone over E with vertex corresponding to the one-dimensional subspace

$$\pi^* \mathrm{H}^0(E, \Omega_E) \subset \mathrm{H}^0(C, \Omega_C)\,.$$

One still has $C \subset \Gamma_\eta$ and the map $\pi : C \to E$ is induced by projecting away from the vertex of Γ_η.

On the other hand, Γ_η is a cubic symmetroid since the two-torsion point η_0 defines a symmetric determinantal equation for $E \subset \mathbb{P}^2$ [Dol12, Section 4.1.3]. But this cubic symmetroid is degenerate in the following sense: when writing

$$\Gamma_\eta = \mathcal{V}\left(\det\left(\sum_{i=0}^{3} A_i x_i\right)\right)$$

with $A_i \in \mathrm{Mat}_{3,3}(k)$ symmetric, then the matrices A_i are linearly dependent. Also, the map c does not exist in the bielliptic case because Γ_η is not a rational surface.

Nevertheless, the linear maps φ, ψ are still defined. Indeed, φ was defined unconditionally. On the other hand, the map

$$\psi : S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta) \to W_\eta \subset S^2\,\mathrm{H}^0(C, \Omega_C)$$

can be defined by taking W_η to be the set of quadrics that are singular at the vertex of Γ_η. The map ψ is then the map sending a conic to the affine cone over it.

All the results from the previous sections still hold in the bielliptic case. However, the proof of Lemma 2.6 needs a separate argument which we give now. Indeed, instead of diagram 2.3 we consider the diagram

$$\begin{tikzcd} & C \arrow[dl, "\phi_\eta"'] \arrow[dr] & \\ \mathbb{P}^2 & & \mathbb{P}^3 \arrow[ll, dashed] \end{tikzcd}$$

where the map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is the projection away from the vertex of Γ_η. The diagram commutes because, as noted above, this projection induces the map $C \to E$. The rest of the proof follows the same line of reasoning as in Lemma 2.6.

2.5. **Tritangent planes and theta derivatives.** In order to compute of the tritangent planes on the left-hand side of Equation (2.4), we require formulas that express them purely in terms of the theta constants. To derive such formulas one begins with the following well-known connection between tritangent planes and theta derivatives.

Theorem 2.11. *Let C be a smooth curve of genus 4 over $\mathbb{C}$ with small period matrix τ.*

For any odd theta characteristic $\begin{bmatrix} a \\ b \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^8/\mathbb{Z}^8$ the equation

$$\sum_{i=1}^{4} \frac{\partial \vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}{\partial z_i}(0,\tau) x_i = 0 \tag{2.5}$$

defines a tritangent plane for the canonical image of C in the $\mathbb{P}^3$ with coordinates $x_1, \ldots, x_4$. Here the basis for $\mathrm{H}^0(C, \Omega_C)$ must be the basis induced by the isomorphism

$$\mathrm{Jac}(C) \cong \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)\,.$$

Proof. See, e.g., [Fio16, p. 5]. □

Next, we want to replace the theta derivatives by theta constants. This is achieved with the same approach as in the genus 3 case [Fio16]: one chooses

5 tritangent planes and applies a coordinate transform that puts them into the standard form

$$x_i = 0, \qquad x_1 + x_2 + x_3 + x_4 = 0.$$

Then Cramer's rule expresses the equations for the other tritangent planes in terms of determinants of Jacobi matrices of theta functions (see [Fio16, Equation (17)] or Equation (3.1) in the present article). The next section explains how these determinants can be expressed in terms of theta constants.

2.6. **The generalized Jacobi derivative identity.** In this section, we introduce Fay's generalization of Jacobi's derivative formula. It expresses the Jacobian determinant of an azygetic system of odd theta functions evaluated at 0 as a polynomial in the theta nullvalues.

First, we will recall some definitions from the theory of theta characteristics. A system of theta characteristics $c_1, c_2, c_3 \in \frac{1}{2}\mathbb{Z}^{2g}$ is called an *azygetic* triple if

$$e_*(c_1 + c_2 + c_3) = -e_*(c_1)e_*(c_2)e_*(c_3)\,,$$

where $e_* : \frac{1}{2}\mathbb{Z}^{2g} \to \{\pm 1\}$ is the parity map $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (-1)^{4a^t b}$. More generally, an arbitrary system of characteristics $c_1, \ldots, c_n$ is called *azygetic* if any triple contained in $c_1, \ldots, c_n$ is an azygetic triple.

A system of theta characteristics is called *essentially independent* if every sum of a subset of even cardinality of the system is non-zero.

Definition 2.12. A *special fundamental system* is a system of $2g+2$ characteristics $m_1, \ldots, m_g, n_1, \ldots, n_{g+2} \in \frac{1}{2}\mathbb{Z}^g$ such that:

i) $m_1, \ldots, m_g, n_1, \ldots, n_{g+2}$ is azygetic.
ii) The characteristics $m_1, \ldots, m_g$ are odd.
iii) The characteristics $n_1, \ldots, n_{g+2}$ are even.

We will now define the lefthand side of the generalized Jacobi derivative identity.

Definition 2.13. Let $m_1, \ldots, m_g \in \frac{1}{2}\mathbb{Z}^{2g}$ be a system of odd characteristics. The *Jacobian nullvalue* of $m_1, \ldots, m_g$ is defined to be the function on the Siegel upper half space

$$D(m_1, \ldots, m_g) : \mathfrak{H}_g \longrightarrow \mathbb{C}$$

given by the formula

$$D(m_1, \ldots, m_g)(\tau) = \pi^{-g} \det \left(\left(\frac{\partial \vartheta[m_i]}{\partial z_j} \right)_{i,j=1,\ldots,g} \right) (0, \tau)\,.$$

Now if $m_1, \ldots, m_g$ is azygetic and essentially independent then, for $g \leqslant 5$, $D(m_1, \ldots, m_g)$ can be expressed as the following polynomial in the theta nullvalues.

Theorem 2.14. *Let $g \leqslant 5$ and $M = \{m_1, \ldots, m_g\}$ be an azygetic essentially independent system of odd theta characterstics. Then for all $\tau \in \mathfrak{H}_g$,*

$$D(m_1, \ldots, m_g)(\tau) = \sum_N \pm \prod_{i=1}^{g+2} \vartheta[n_i](0, \tau) \tag{2.6}$$

where the sum runs over all sets $N = \{n_1, \ldots, n_{g+2}\}$ such that $M \cup N$ is a special fundamental system.

Furthermore, the signs $\pm$ are explicit, unique and independent of τ.

Proof. See [Fro85] for $g = 4$ and [Fay79] for $g \leqslant 5$. □

Remark 2.15. The number of terms on the right of Fay's generalized Jacobi derivative identity is given by the following table.

g	1	2	3	4	5
no. of terms	1	1	1	2	8

Remark 2.16. Igusa [Igu80][Igu83] has a conjectural generalization of Fay's generalized Jacobi identity to arbitrary $g \geqslant 6$. However, as it is known that in this case a Jacobian nullvalue $D(m_1, \ldots, m_g)$ cannot be a polynomial in the theta constants [Fay79, p. 12], Igusa's conjectural formula has a sum of Jacobian nullvalues on the left-hand side.

It would be an interesting problem to use Igusa's conjectural identity to express Jacobian nullvalues as rational functions in the theta constants. Such expressions must exist by general principles [Igu72, Theorem V.9], but they seem to be unknown.

3. Reconstructing the curve

3.1. **An auxiliary set of odd theta characteristics.** In order to use the generalized Jacobi identity (Theorem 2.14) for the computation of the pairs of tritangent planes in Equation (2.4) we will need a set of odd theta characteristics satisfying certain properties. Indeed, on the left-hand side of the identity we have the Jacobian nullvalue of a set of four azygetic essentially independent odd theta characteristics. Trying to optimize the use of the generalized Jacobi identity leads us to choose the following auxiliary set of odd theta characteristics.

Lemma 3.1. *Let $\eta \in \frac{1}{2}\mathbb{Z}^8/\mathbb{Z}^8$ be arbitrary. There exist odd theta characteristics $\xi_1, \ldots, \xi_5$, $\chi_1, \chi_1', \ldots, \chi_{10}, \chi_{10}' \in \frac{1}{2}\mathbb{Z}^8/\mathbb{Z}^8$ such that*

i) *$\xi_1, \ldots, \xi_5$ is azygetic and essentially independent.*

ii) *For all $i \in \{1, \ldots, 10\}$ the systems $\xi_1, \ldots, \xi_4, \chi_i$ and $\xi_1, \ldots, \xi_4, \chi_i'$ are azygetic and essentially independent.*

iii) *For all i one has $\chi_i - \chi_i' = \eta$.*

Proof. Without loss of generality we can assume that $\eta = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. (This is the convention used in, e.g., [FR70].) An explicit answer is then given by:

$$\xi_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \xi_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \xi_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\xi_4 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \xi_5 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\chi_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \chi_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \chi_3 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$\chi_4 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \chi_5 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \chi_6 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$
$$\chi_7 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \chi_8 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \chi_9 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$
$$\chi_{10} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

and $\chi'_i = \chi_i + \eta$ for $i \in \{1, \dots, 10\}$. ☐

Remark 3.2. The answer in the previous proof is found by choosing $\xi_1, \dots, \xi_5$ suitably and then solving a system of equations over $\mathbb{F}_2$ for $\chi_1, \chi'_1, \dots, \chi_{10}, \chi'_{10}$ allowing η to be arbitrary. In the end one transforms the characteristics such that η becomes $\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

One can still see the (affine) linear structure in the shape of the characteristics χ_i, χ'_i. Indeed, the χ_i are all the odd theta characteristics whose left 2×2 block is
$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$
From this observation one sees that one cannot add further pairs χ_i, χ'_i with the required properties to the system.

For the rest of the article, we define $H_i, H'_i \in \mathrm{H}^0(C, \Omega_C)$ to be the linear forms cutting out the tritangent planes corresponding to χ_i, χ'_i. By construction, for any $i = 1, \dots, 10$ the tritangent planes H_i and H'_i differ by η. We denote by $\ell_i \in \mathrm{H}^0(C, \Omega_C \otimes \eta)$ the linear form cutting out the bitangent of X that corresponds to (H_i, H'_i) under Milne's bijection (Theorem 2.8).

The main point of the special conditions in Lemma 5.1 is that they guarantee that the tritangent planes H_i, H'_i can be written with simple formulas in terms of the theta constants, as we now explain. Indeed, we choose the coordinate system for $\mathbb{P}^3$ such that the tritangent planes corresponding to the odd characteristics ξ_i, $i = 1, \dots, 5$ are in normal form
$$x_i = 0, \quad i \in \{0, \dots 3\},$$

$$x_0 + x_1 + x_2 + x_3 = 0\,.$$

Then, with the same argument as in [Fio16, Equation (17)] one sees that for any $i \in \{1, \dots, 10\}$ the tritangent plane H_i is given by the equation

$$\begin{aligned} \frac{D(\chi_i, \xi_2, \xi_3, \xi_4)}{D(\xi_5, \xi_2, \xi_3, \xi_4)} x_0 + \frac{D(\xi_1, \chi_i, \xi_3, \xi_4)}{D(\xi_1, \xi_5, \xi_3, \xi_4)} x_1 + & \\ \frac{D(\xi_1, \xi_2, \chi_i, \xi_4)}{D(\xi_1, \xi_2, \xi_5, \xi_4)} x_2 + \frac{D(\xi_1, \xi_2, \xi_3, \chi_i)}{D(\xi_1, \xi_2, \xi_3, \xi_5)} x_3 &= 0 \end{aligned} \tag{3.1}$$

in this coordinate system. An analogous formula with χ_i replaced by χ_i' gives the tritangent planes H_i'.

By Lemma 3.1 the set $\xi_1, \dots, \xi_5$ and all the sets $\xi_1, \dots, \xi_4, \chi_i$ as well as $\xi_1, \dots, \xi_4, \chi_i'$ for $i = 1, \dots, 10$ are azygetic and essentially independent. This implies that the Jacobian nullvalue in our equations for H_i, H_i' can be expressed in terms of theta constants via Theorem 2.14.

Remark 3.3. The denominators in Equation (3.1) are non-zero for a generic curve. But it can happen that they vanish: such curves must exist because the corresponding locus in the Satake compactification of the Siegel moduli space is given by the vanishing locus of a non-zero polynomial in the theta constants on the (open) Torelli locus. By [FC14, Theorem 2.3] such a polynomial is a section of an ample line bundle. Furthermore, since the boundary of the Torelli locus in the Satake compactification has codimension two, this locus must be non-empty. Nevertheless, the authors do not know an explicit example of this phenomenon.

3.2. **Finding the quadric.** In this section, we describe how to recover the quadric. The first step is obtaining the equation for the map φ defined in the diagram (2.2). To do this, we construct and solve a linear system as follows.

Recall from Theorem 2.8 that

$$\varphi(HH') = \lambda \ell^2$$

for any pair of tritangent planes (H, H') that maps to a bitangent ℓ under Milne's bijection.

Lemma 3.4. *For a generic genus* 4 *curve, the elements* $H_i H_i' \in V_{C,\eta}$*, for* $i = 1, \dots, 10$ *form a generating set of* $V_{C,\eta}$.

Proof. Since the property is open in the moduli space and the moduli space is irreducible, it suffices to show that one genus 4 curve satisfies the condition of the lemma. We will give an example in the proof of Lemma 3.5 below. □

Using the knowledge of the equations of the pairs of tritangent planes H_i, H_i' and the corresponding bitangents ℓ_i for $i = 1, \dots, 10$, we know everything in the equations

$$\varphi(H_i H_i') = \lambda_i \ell_i^2,\ i = 1, \dots, 10 \tag{3.2}$$

except for the scalars λ_i. The linear dependencies satisfied by the H_iH_i' yield a homogeneous linear system of equations with unknowns λ_i. (Recall that $H_iH_i' \in V_{C,\eta}$ and $\dim(V_{C,\eta}) = 7$, as mentioned in Section 2.2.) Solving this linear system, we recover the linear map φ, and thus the quadric Q from $\ker(\varphi)$.

Lemma 3.5. . *For a generic genus 4 curve the vector $(\lambda_i)_{1,\dots,10}$ is uniquely determined from this linear system up to a scalar.*

Proof. With the same argument as in Lemma 3.4 it suffices to check one example.

We will now construct an example over $\mathbb{F}_{37}$ and verify the required properties with the aid of Magma. (See the files `rational-trits.m` and `illustration.m` in [HPS24c].) Consider the plane quartic $X \subset \mathbb{P}^2$ with equation

$$X : t^4 + 14t^3u + 16t^3v + 32t^2u^2 + 26t^2uv + 18t^2v^2 + 29tu^3 + 4tu^2v + 11tuv^2 + 2tv^3 + 26u^4 + 16u^3v + 27u^2v^2 + 22uv^3 + 11v^4 = 0.$$

Using Recillas's trigonal construction [Rec93, Theorem 2.15] with the explicit geometric version of [BS20, Theorem 1.5] one can compute a genus 4 curve C with a two-torsion point η such that $\mathrm{Jac}(X)$ is isomorphic to the Prym. All the explicit properties of the curve C that we claim below will be a consequence of this construction.

We find that the canonical embedding of the curve C into $\mathbb{P}^3$ is cut out by the equations

$$\begin{aligned} 0 &= 11xw + yz, \\ 0 &= x^3 + 3x^2y + 9x^2z + 15x^2w + 23xy^2 + 10xyw + 12xz^2 + 6xzw + 12xw^2 \\ &\quad + 8y^3 + y^2w + 17yw^2 + 11z^3 + 14z^2w + 7zw^2 + 28w^3 \,. \end{aligned}$$

C has the following 10 products of pairs of tritangent planes (H_i, H_i') differing by η. (Note that many of these are irreducible quadrics over $\mathbb{F}_{37}$, but factor as a product of two linear forms over $\mathbb{F}_{37^2}$.)

$$\begin{aligned} H_1H_1' &= 2x^2 + 9xy + 18xz + 6xw + 8y^2 + 20yz + 5yw + 15z^2 + 23zw + 33w^2, \\ H_2H_2' &= 21x^2 + 25xy + 20xz + 29xw + 2y^2 + 32yz + 34yw + 16z^2 + 5zw + 36w^2, \\ H_3H_3' &= 4x^2 + 13xy + 28xz + 9xw + 7y^2 + 28yz + 22yw + 19z^2 + 20zw + 33w^2, \\ H_4H_4' &= 30x^2 + 27xy + 6xz + 29xw + 28y^2 + 4yz + 16yw + 25z^2 + 4zw + 13w^2, \\ H_5H_5' &= 24x^2 + 22xy + 2xz + 32xw + 22y^2 + 4yz + 27yw + 31zw + 21w^2, \\ H_6H_6' &= 9x^2 + 16xy + 19xz + 6xw + 17y^2 + 12yz + 14yw + 23z^2 + 15zw + 32w^2, \\ H_7H_7' &= 29x^2 + 26xy + 30xz + 36xw + 22y^2 + 34yz + 11yw + 12z^2 + 35zw + 23w^2, \\ H_8H_8' &= 25x^2 + 26xy + 30xz + 24xw + 32y^2 + 16yz + 2yw + 28z^2 + 3zw + 33w^2, \\ H_9H_9' &= 8x^2 + 30xz + 17xw + 19y^2 + 31yz + 3yw + 23z^2 + 17w^2, \\ H_{10}H_{10}' &= 13x^2 + 34xy + 28xz + 14y^2 + 13yz + 22yw + 29z^2 + 16zw + 14w^2 \,. \end{aligned}$$

(It is important to ensure that the odd theta characteristics corresponding to (H_i, H_i') are given by the (χ_i, χ_i') from Lemma 3.1.)

By explicit computation one checks that the 7 quadratic forms $H_3H_3', \dots, H_{10}H_{10}'$ are linearly independent and thus form a basis of $V_{C,\eta}$. This proves Lemma 3.4.

Next, the bitangent lines of X corresponding to (H_i, H_i') under Milne's bijection are given by

$$\begin{aligned}
\ell_1 &= 12t + 16u + 23v,\\
\ell_2 &= 3t + 19u + 10v,\\
\ell_3 &= t + u + v,\\
\ell_4 &= 10t + 35u + 23v,\\
\ell_5 &= 7t + 17u + 24v,\\
\ell_6 &= 23t + 27u + 30v,\\
\ell_7 &= 24t + 27u + 31v,\\
\ell_8 &= t,\\
\ell_9 &= 35t + 19u + v,\\
\ell_{10} &= 13t + 17u + 23v\,.
\end{aligned}$$

To recover φ, we now use the Equations (3.2) $\varphi(H_iH_i') = \lambda_i \ell_i^2$, which imply that the row vector $\Lambda = (\lambda_i)$ must satisfy the system of linear equations $\Lambda M = 0$ where M is the matrix below.

$$\begin{pmatrix}
33 & 14 & 34 & 34 & 33 & 11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 9 & 3 & 23 & 28 & 10 & 26 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 2 & 1\\
19 & 22 & 6 & 20 & 21 & 18 & 2 & 14 & 24 & 6 & 10 & 35 & 10 & 33 & 9 & 30 & 13 & 27\\
29 & 14 & 35 & 17 & 11 & 23 & 19 & 13 & 14 & 29 & 34 & 24 & 31 & 29 & 17 & 22 & 36 & 8\\
11 & 21 & 11 & 26 & 29 & 12 & 10 & 9 & 10 & 27 & 23 & 21 & 4 & 11 & 4 & 33 & 24 & 1\\
26 & 3 & 24 & 4 & 27 & 34 & 35 & 14 & 1 & 31 & 15 & 23 & 36 & 7 & 19 & 34 & 26 & 30\\
21 & 0 & 0 & 0 & 0 & 0 & 22 & 0 & 0 & 0 & 0 & 0 & 29 & 0 & 0 & 0 & 0 & 0\\
4 & 35 & 33 & 28 & 1 & 1 & 21 & 8 & 16 & 36 & 33 & 33 & 32 & 21 & 5 & 2 & 8 & 8\\
8 & 1 & 34 & 22 & 16 & 13 & 1 & 14 & 32 & 12 & 2 & 34 & 7 & 24 & 2 & 10 & 14 & 16
\end{pmatrix}$$

One readily verifies that the left kernel of M is one-dimensional and generated by $(1, 21, 13, 32, 8, 33, 17, 3, 19, 18)$. This proves Lemma 3.5. $\square$

For illustrational purposes we continue to explain how the quadric containing C is recovered. The matrix for

$$\varphi : V_{C,\eta} \longrightarrow S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta)$$

is given by

$$\begin{pmatrix} 18 & 22 & 30 & 24 & 3 & 2 & 8 \\ 15 & 17 & 27 & 17 & 0 & 36 & 1 \\ 31 & 24 & 30 & 25 & 0 & 35 & 34 \\ 17 & 18 & 7 & 35 & 0 & 14 & 22 \\ 16 & 16 & 32 & 5 & 0 & 19 & 16 \\ 19 & 20 & 26 & 20 & 0 & 19 & 13 \end{pmatrix}$$

with respect to the basis $H_3H_3', \dots, H_{10}H_{10}'$ on $V_{C,\eta}$ and the standard (monomial) basis on $S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta)$. By computing the kernel of φ one recovers the quadratic form

$$27xw + 26yz = 26(11xw + yz)$$

which vanishes on C.

3.3. **Finding the cubic.** In this section, we explain how the knowledge of the map

$$\varphi : V_{C,\eta} \longrightarrow S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta)$$

can be used to recover the equations of the curve C. As we know that φ recovers Q, it remains to explain the calculation of the cubic. We begin by giving a connection between the natural right-inverse ψ of φ and the Cayley cubic Γ_η. For this purpose, we will introduce a sextic form G which vanishes with multiplicity two on Γ_η. In this definition we use the abbreviations $V = \mathrm{H}^0(C, \Omega_C)$, $W = \mathrm{H}^0(C, \Omega_C \otimes \eta)$.

Let G be the homogeneous form that is given by the following composition of polynomial maps

$$V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to V_{C,\eta}^\vee \xrightarrow{\psi^\vee} (S^2W)^\vee \cong S^2(W^\vee) \xrightarrow{\mathrm{disc}} k\,, \tag{3.3}$$

where the first map is $x \mapsto x \otimes x$ and the map $(S^2V)^\vee \to V_{C,\eta}^\vee$ is dual to the defining inclusion.

It is easy to see that G has degree 6 because the first map has degree 2, the map disc has degree $\dim(W) = 3$, and all the other maps are linear.

The next proposition will show that the homogeneous form G is the square of a cubic form cutting out the cubic symmetroid.

Remark 3.6. The intuition behind considering the composition

$$V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to V_{C,\eta}^\vee \xrightarrow{\psi^\vee} (S^2W)^\vee \cong S^2(W^\vee)$$

is as follows: we want to understand the image Γ_η of the map

$$c : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3\,.$$

But we do not have access to the map c explicitly; we only have the map φ which is linked to the way c transforms quadrics. The best we can do starting from a point $x \in \mathbb{P}^3$ is to form the subvector space of quadrics vanishing at x and, then, apply the map φ. This gives us a linear system of conics on $\mathbb{P}^2$ from which we can hope to detect whether or not x is in the image of c. Dualizing this consideration shows that it is natural to study the composition

$$V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to V_{C,\eta}^\vee \xrightarrow{\psi^\vee} (S^2W)^\vee \cong S^2(W^\vee)\,.$$

Proposition 3.7. *One has $\mathcal{V}(G) = 2\Gamma_\eta$.*

Proof. Assume first that (C,η) is not bielliptic. We begin by showing that G vanishes on Γ_η. Let $U \subset \Gamma_\eta$ be the complement of the base locus of the birational map $c^{-1} : \Gamma_\eta \dashrightarrow \mathbb{P}^2$. We will show that G vanishes on the open subset U. To that end, let $x \in U$ be arbitrary and let $y = c^{-1}(x) \in \mathbb{P}^2$ denote its image. Then x (resp., y) gives a non-zero vector in $V^\vee$ (resp., $W^\vee$). We claim that under the composition

$$\gamma : V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to V_{C,\eta}^\vee \xrightarrow{\psi^\vee} (S^2W)^\vee \cong S^2(W^\vee)\,,$$

x maps to a multiple of $y \otimes y$.

Recall that W_η denotes the image of ψ. Then the map γ factors as

$$V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to W_\eta^\vee \xrightarrow{\psi^\vee} (S^2W)^\vee \cong S^2(W^\vee)$$

Next, we know that the map ψ is given by pull-pushing quadrics through the diagram

$$\begin{tikzcd} & & S \arrow[dll] \arrow[dr] & \\ \mathbb{P}(\mathrm{H}^0(C,\Omega_C\otimes\eta)) \cong \mathbb{P}^2 \arrow[rrr, dashed, "c"] & & & \Gamma_\eta \end{tikzcd}\ ,$$

i.e., for every $q \in S^2W = S^2\,\mathrm{H}^0(C,\Omega_C\otimes\eta)$ the quadratic form $\psi(q)$ is the unique quadratic form vanishing on the curve $c(\mathcal{V}(q)) \subset \Gamma_\eta$. Therefore, $\gamma(x)$ must be orthogonal to the quadratic forms vanishing at y and thus is a multiple of $y \otimes y$. (It could possibly be zero.) This proves the claim.

Next, the observation $\mathrm{disc}(y \otimes y) = 0$ shows that G vanishes on Γ_η.

To prove that G vanishes with multiplicity two, one can use the same argument with adjugate matrices as in the discussion preceding Theorem 4.1.4 in [Dol12] to show that $\gamma_\eta^3 \mid G^2$ where γ_η is a cubic form cutting out Γ_η. This implies that $\gamma_\eta^2 \mid G$ because Γ_η is irreducible.

It remains to show that $G \neq 0$. One possible argument would be to go through the classification of symmetroid cubic surfaces on [Cat81, §1, p. 33] and explicitly verify that $G \neq 0$ in every case. Alternatively, we propose the following general proof. Choose a plane $H \subset \mathbb{P}^3$ such that $\Gamma_\eta \cap H$ is smooth. Then $\Gamma_\eta \cap H$ is a smooth plane curve and the restriction $\gamma_{\eta|H}$ gives a determinantal equation for it. Hence, the general theory of determinantal equations for smooth plane curves from Dolgachev's book [Dol12, Section 4.1.2] applies. The degree six form $G_{|H}$ appears as a special case of the construction preceding [Dol12, Theorem 4.1.4] (where it is denoted by $\det(N)$). In loc. cit. he proves that this polynomial is non-zero. Therefore, we conclude that $G_{|H} \neq 0$ and thus $G \neq 0$. This proves the non-bielliptic case.

Assume now that (C, η) is bielliptic. Then Γ_η is a cone over a smooth plane cubic $E \subset \mathbb{P}^2$. Furthermore, it is easy to see that $\mathcal{V}(G)$ is a cone over the same vertex. Thus we are reduced to a statement about the symmetric determinantal representation of E in $\mathbb{P}^2$. Then Dolgachev's argument applies directly and the proposition follows. $\square$

However, we do not have direct access to the map ψ in our reconstruction method. The next lemma shows how to find a cubic vanishing on C from the knowledge of φ alone.

Lemma 3.8. *Let $\widetilde{\psi}$ be any right inverse of φ. Define $\widetilde{G}$ using the same formula as for G in (3.3) but with ψ replaced by $\widetilde{\psi}$. Then the congruence*

$$G \equiv \widetilde{G} \mod Q$$

holds true.

Proof. By Lemma 2.6, the map ψ satisfies $\varphi \circ \psi = \mathrm{id}$ after fixing the ambiguity in the definition. Then $\psi, \widetilde{\psi}$ are two right inverses of φ. Therefore the difference $\psi - \widetilde{\psi}$ satisfies

$$\mathrm{im}(\psi - \widetilde{\psi}) \subseteq \ker(\varphi) = \mathrm{span}(Q)\,.$$

Dually, the difference $\psi^\vee - \widetilde{\psi}^\vee : V_{C,\eta}^\vee \longrightarrow (S^2W)^\vee$ vanishes on $\mathrm{span}(Q)^\perp \subset V_{C,\eta}^\vee$.

We are now ready to show that $G \equiv \widetilde{G} \mod Q$. We claim that the two polynomial maps defining $G, \widetilde{G}$ agree when restricted to the vanishing locus of Q. Indeed, let $x \in V^\vee$ be an arbitrary vector satisfying $Q(x) = 0$. The latter means that the element $x \otimes x \in S^2(V^\vee) \cong (S^2V)^\vee$ is orthogonal to Q. Therefore under the first arrows in the composition defining $G, \widetilde{G}$

$$V^\vee \to S^2(V^\vee) \cong (S^2V)^\vee \to V_{C,\eta}^\vee$$

x maps into $\mathrm{span}(Q)^\perp$. Above we noted that $\psi^\vee$ and $\widetilde{\psi}^\vee$ agree on $\mathrm{span}(Q)^\perp$. This is sufficient to prove the claim. Therefore the two polynomial maps defining $G, \widetilde{G}$ agree when restricted to the vanishing locus of Q. This implies that

$$G \equiv \widetilde{G} \mod Q\,.$$

$\square$

Using the previous lemma we can find a cubic Γ vanishing on C by extracting a square root of $\widetilde{G}$ mod Q. Furthermore, one has $C = Q \cap \Gamma_\eta = Q \cap \Gamma$.

3.4. **Main result.** Summarizing the discussion of the previous sections, we have proven the following theorem.

Theorem 3.9. *Let C be a generic genus 4 curve over an algebraically closed field k of characteristic 0 or of characteristic p with p large enough. There are explicit formulas for the equations of C in terms of its theta constants. They use only the following algebraic operations: Elementary arithmetic, taking square roots, and solving linear systems of equations.*

We now give an overview of these formulas:

Algorithm 3.1: Reconstruct a genus 4 curve from its theta constants

---

Input : Theta constants $\vartheta_C\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ of a smooth non-hyperelliptic genus 4 curve C defined over k.

Output: A quadric Q and a cubic Γ in $\mathbb{P}^3_k$ such that $C \cong Q \cap \Gamma$ or an error if C is not generic enough.

1 Choose the two-torsion point given by $\eta = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

2 Compute the theta constants for the plane quartic X via the Schottky-Jung relations

$$\vartheta_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^2 = \vartheta_C\left[\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix}\right]\vartheta_C\left[\begin{smallmatrix} 0 & a \\ 1 & b \end{smallmatrix}\right].$$

Compute the 10 bitangent lines ℓ_i via the Aronhold-Weber formulas [Dol12, Theorem 6.1.9] [Fio16, Theorem 2] from the theta constants $\vartheta_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. (Make sure to take the bitangents with the correct odd theta characteristics.)

3 Try to compute the tritangent planes H_i, H_i' from the Equations (3.1). If there is a division by zero, give an error.

4 Verify that the H_iH_i' generate a linear space of dimension 7 or throw an error.

5 From the Equations (3.2)

$$\varphi(H_iH_i') = \lambda_i \ell_i^2$$

we get a homogeneous linear system of equations for the λ_i.

6 Verify that the kernel of this linear system of equations is one-dimensional or give an error.

7 From the λ_i compute a matrix for the linear map φ.

8 Compute a generator for $\ker(\varphi)$ and call it Q.

9 Choose a linear map $\widetilde{\psi}$ such that $\varphi \circ \widetilde{\psi} = \mathrm{id}$.

10 Compute the degree six polynomial $\widetilde{G}$ as in Lemma 3.8.

11 Compute a square root Γ of $\widetilde{G}$ mod Q.

12 Return Q and Γ.

Remark 3.10. When using the Schottky-Jung relations one takes a square root to compute the $\vartheta_X\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. This leads to a sign choice that has to be made correctly. For this one can use the discussion following [Gla80, Theorem 3.1] where Glass makes use of the quartic Riemann identities to find a correct choice of signs. At this point, the assumption that the $\vartheta_C\big[\begin{smallmatrix}a\\b\end{smallmatrix}\big]$ are the theta constants of a genus 4 curve is used. Indeed, the latter implies that the Schottky modular form vanishes. By construction of this modular form, this is equivalent to requiring that the $\vartheta_X\big[\begin{smallmatrix}a\\b\end{smallmatrix}\big]$ satisfy the quartic Riemann identities.

Remark 3.11. If one of the $\vartheta_C\big[\begin{smallmatrix}0&a\\0&b\end{smallmatrix}\big]$ or $\vartheta_C\big[\begin{smallmatrix}0&a\\1&b\end{smallmatrix}\big]$ vanish, then the Schottky-Jung relations imply that X has a vanishing even theta-null. This means that the curve X becomes hyperelliptic. Our method still works in this case by making the following adjustment: the canonical map $X \to \mathbb{P}^2$ is a degree two cover of a smooth conic ramified in eight points. Replace the 28 bitangent lines by the $\binom{8}{2} = 28$ lines through pairs of ramification points (see also Remark 2.9).

Notice that in this situation, the curve C has a vanishing theta null and thus the canonical quadric Q becomes singular. Conversely, if C has a vanishing theta null, we can achieve that $\vartheta_C\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big] = 0$ by choosing the level structure appropriately. Thus we can arrange for the curve X to be hyperelliptic.

In computations over $\mathbb{C}$ it is advantageous to exploit this because using a hyperelliptic curve X will increase the speed and minimize the numerical error.

Remark 3.12. There are three possibilities for the curve C to not be generic enough: (i) there is a division by zero in step 3, (ii) the products H_iH_i' do not generate $V_{C,\eta}$, or (iii) the λ_i are not uniquely determined by the linear system of equations in step 6. The first situation can occur; see Remark 3.3. However, the authors do not know if there exist curves satisfying one of the other two phenomena. Using the same strategy as in the proof of Lemma 3.4, 3.5, i.e., by exhibiting an example over a finite field, we obtained that the open locus in the moduli space given by conditions (ii) and (iii) has non-empty intersection with the following loci:

- C has a vanishing even theta constant, i.e., the quadric Q is singular;
- Any degeneration of the cubic symmetroid Γ_η;
- In particular, the bielliptic locus.

Anyway, if the reconstruction fails for one of the three reasons above, all hope is not lost—one can choose a different 2-torsion point η and try again. We conjecture that for every non-hyperelliptic genus 4 curve over a field of characteristic $\neq 2$ there exists a two-torsion point $\eta \in \mathrm{Jac}(C)[2] \setminus \{0\}$ such that our reconstruction method works.

4. Examples

Example 4.1. Gluing. Consider the hyperelliptic genus 2 curve

$$C_1 : y^2 = 24x^5 + 36x^4 - 4x^3 - 12x^2 + 1$$

which has LMFDB label `20736.l.373248.1`. From the information on the curve's homepage, we see that the geometric endomorphism algebra of $\mathrm{Jac}(C_1)$ is the quaternion algebra $B_{2,3}$, i.e., the unique quaternion algebra over $\mathbb{Q}$ ramified at 2 and 3. We look for interesting genus two curves C_2 such that the quotient of $A = \mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2)$ by a maximal isotropic subgroup of $A[2]$ is the Jacobian of a smooth genus 4 curve C (possibly up to a quadratic twist). Using the criterion of [BK23, Theorem 1.2] one finds that, for example, the curve

$$C_2 : y^2 = 3x^5 - 68x^4 + 159x^3 + 232x^2 - 132x + 16$$

has this property. Using the methods of Costa–Mascot–Sijsling–Voight [CMSV19], we verify that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(\mathrm{Jac}(C_2)) \cong \mathbb{Q}(\sqrt{5})$.

Our method produces the following equations for the genus 4 curve C:

$$\begin{aligned} 0 &= 10x^2 + 8xy - 9y^2 - 33z^2 - 30zw - 40w^2 \\ 0 &= -6x^3 - 2x^2y - xy^2 + 5xz^2 - 22xzw - 5xw^2 + 3y^3 + 11yz^2 \\ &\quad + 10yzw - 11yw^2 \,. \end{aligned}$$

By construction we have that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(\mathrm{Jac}(C)) \cong B_{2,3} \times \mathbb{Q}(\sqrt{5})$.

A priori the curve will come out in a coordinate system where the equations are not defined over $\mathbb{Q}$. Using the knowledge of a big period matrix for $\mathrm{Jac}(C)$ associated to a $\mathbb{Q}$-rational basis for $\mathrm{H}^0(C, \Omega_C)$, we found the PGL_4 transformation that changes coordinates into this $\mathbb{Q}$-rational basis.

Example 4.2. Modular Jacobians. Let f be the modular form orbit with LMFDB label 778.2.a.a. We use our method to compute the abelian fourfold that corresponds to it via modularity for RM abelian varieties over $\mathbb{Q}$. This abelian variety A is the subvariety of $J_0(778) = \mathrm{Jac}(X_0(778))$ corresponding to a 4-dimensional Hecke-invariant subspace. It has RM by the field of Hecke eigenvalues which is the totally real subfield of $\mathbb{Q}(\zeta_{15})$; this is the quartic field with LMFDB label 4.4.1125.1. We begin by computing its period matrix using Magma's command `Periods`. However, this period matrix need not correspond to a principally polarized abelian variety: the pullback of the principal polarization on $J_0(778)$ is not in general principal.

To remedy this, we call the command `SomePrincipalPolarizations` from the ModularCurves GitHub repository [S⁺23]. This produces several big period matrix candidates. Evaluating the Schottky modular form on the first of these yields a complex number with absolute value $6.2124 \cdot 10^{-300}$ when calculated with precision 300, indicating that this principally polarized abelian variety is likely a Jacobian. Applying our method then produces the corresponding genus 4 curve, which, after a change of variable, is isomorphic to the curve C in $\mathbb{P}^3$ given by the following equations.

$$\begin{aligned}
0 &= x^2 - xz - xw - y^2 - yw + 2z^2 + zw - 4w^2 \\
0 &= 2xyw - 2xz^2 - 12xzw - 10xw^2 - y^2z - 2y^2w + yzw + 4yw^2 + 2z^3 \\
&\quad - 20zw^2 - 18w^3
\end{aligned}$$

As a sanity check for this heuristic example, we compute the places of bad reduction of C. Using the methods from [Bou23], we compute the invariants of C and in particular the discriminant. The result is $2^9 \cdot 113^{30} \cdot 389^4$ and therefore candidates for the primes of bad reduction are $2, 113$, and 389. But, modulo 113 the given equations reduce to a smooth genus 4 curve, although the canonical quadric degenerates to a cone. Thus the primes of bad reduction are 2 and 389 as one would expect, since the level of f is $778 = 2 \cdot 389$. As a further sanity check, we have computed and compared the local L-factors of f and C for primes up to 1000 and verified that they match. A priori the Jacobian of C could have been a quadratic twist of the abelian variety corresponding to f, but our computations also showed this is not the case.

We thank Noam Elkies for suggesting this example, as well as Edgar Costa for his help in computing polarizations.

5. Future work

5.1. **Constructing CM curves.** Abelian varieties with complex multiplication are another source of examples where the period matrix is known, but the actual equations defining the varieties are not. Furthermore, there are interesting conjectures about when a CM p.p.a.v. variety is a Jacobian. Indeed, for certain values of n, a CM p.p.a.v. is automatically the Jacobian of a degree n superelliptic curve if the CM order contains a primitive n^{th} root of unity and certain conditions on the CM-type are satisfied, as shown in the work of de Jong and Noot [DJN91]. For example, if $n = 3$ and $g = 4$, then they require that the CM order contains a primitive third root of unity, say ζ_3. Furthermore, the action of ζ_3 on the differential forms of the abelian variety must be the same as the action of the automorphism $y \mapsto \exp\left(\frac{2\pi i}{3}\right) y$ on the differential forms of a superelliptic curve of the form

$$y^3 = x(x-1)(x-\lambda)(x-\mu)(x-\nu)\,. \tag{5.1}$$

This can be translated into requiring that the CM type $\Phi = \{\varphi_1, \ldots, \varphi_4\}$ satisfies

$$\begin{aligned}
\varphi_1(\zeta_3) &= \exp\left(\frac{2\pi i}{3}\right) \\
\varphi_2(\zeta_3) &= \exp\left(\frac{2\pi i}{3}\right)^2 \\
\varphi_3(\zeta_3) &= \exp\left(\frac{2\pi i}{3}\right)^2 \\
\varphi_4(\zeta_3) &= \exp\left(\frac{2\pi i}{3}\right)^2 .
\end{aligned} \tag{5.2}$$

The family 5.1 and the other families of superelliptic curves from [DJN91] have the special property that the dimension of the family (3, in the above example) matches the dimension of a PEL-Shimura subvariety where the endomorphism structure is given by $\mathbb{Z}[\zeta_n]$ with a certain tangent space action. (In our example this action would be given by the same conditions as in (5.2).) From this they conclude that the CM-points of that Shimura variety give rise to infinitely many curves with CM Jacobian. Moonen [Moo10] has shown that the collection of special (in the above sense) families of superelliptic curves is finite and listed all of them.

To rule out CM fields containing primitive roots of unity, Moonen and Oort [MO11] defined the notion of a Weyl CM field. These are CM fields K of degree $[K : \mathbb{Q}] = 2g$ such that the Galois group of the splitting field is as large as possible, i.e., $(\mathbb{Z}/2)^g \rtimes S_g$. They put forward the conjecture that for a given fixed $g \geqslant 4$ there are finitely curves of genus g whose Jacobian has CM by a Weyl CM field.

From this perspective it would be interesting to compute explicit examples of genus 4 curves with CM.

5.2. **Higher genus.** In work in progress [HPS24b] we plan to generalize the Milne correspondence to arbitrary g. This would give a bijection between pairs of odd theta hyperplanes in $\mathbb{P}^{g-1}$ for C and odd theta hyperplanes in $\mathbb{P}^{g-2}$ for the Prym variety $\mathrm{Prym}(C, \eta)$. Notice that in this context the Prym variety is no longer automatically a Jacobian. Nevertheless, one can define odd theta hyperplanes for arbitrary p.p.a.v.s; however, these will lack the geometric interpretation as multitangents of a canonically embedded curve (unless the p.p.a.v. happens to be a Jacobian).

This generalization would give formulas for the equations of a generic genus 5 curve in terms of its theta constants. Indeed, for $g \geqslant 5$ the image of the canonical embedding of a generic curve of genus g is cut out by quadratic equations. For $g = 5$, one should be able to compute these equations with a suitable adaption of the results in Section 3.2.

For $g = 6$ we also expect that our method works. However, in order to get a formula purely in terms of theta constants, it remains to find a formula for the Jacobian nullvalues as a rational function in the theta constants; see also Remark 2.16.

For $g \geqslant 7$ another challenge appears: there will be quadratic forms vanishing on the image of the Prym canonical map

$$\phi_\eta : C \longrightarrow \mathbb{P}\left(\mathrm{H}^0(C, \Omega \otimes \eta)\right) \cong \mathbb{P}^{g-2}\,.$$

This means that we can no longer view $V_{C,\eta}$ as a linear subspace of $S^2\,\mathrm{H}^0(C, \Omega_C)$ because here the injectivity of the map

$$S^2\,\mathrm{H}^0(C, \Omega_C \otimes \eta) \to \mathrm{H}^0(C, \Omega_C^{\otimes 2})$$

in (2.2) was crucially used. Nonetheless, one can still use odd theta data to produce a system of equations satisfied by the analogues of the unknown constants λ_i from Equation (3.2), but these equations will be quadratic instead of linear.

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