

GLUING CURVES OF GENUS 1 AND 2 ALONG THEIR 2-TORSION

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ABSTRACT. Let X (resp. Y) be a curve of genus 1 (resp. 2) over a base field k whose characteristic does not equal 2. We give criteria for the existence of a curve Z over k whose Jacobian is up to twist $(2, 2, 2)$ -isogenous to the products of the Jacobians of X and Y . Moreover, we give algorithms to construct the curve Z once equations for X and Y are given. The first of these is based on interpolation methods involving numerical results over \mathbb{C} that are proved to be correct over general fields a posteriori, whereas the second involves the use of hyperplane sections of the Kummer variety of Y whose desingularization is isomorphic to X . As an application, we find a twist of a Jacobian over \mathbb{Q} that admits a rational 70-torsion point.

INTRODUCTION

One of the most fundamental properties of abelian varieties is their unique decomposition up to isogeny, also called Poincaré’s Complete Reducibility Theorem [5, 5.3.7]: An abelian variety A over a field k is isogenous to a product

$$A \sim B_1^{e_1} \times \cdots \times B_n^{e_n} \tag{0.1}$$

where the abelian varieties B_i are simple and pairwise non-isogenous over k , and this decomposition is unique in the sense that up to reordering the isogeny classes of the abelian varieties B_i and the corresponding exponents e_i are uniquely determined.

When $A = \text{Jac}(Z)$ is the Jacobian of a curve of small genus, then there exist algorithms after [12] to calculate the decomposition (0.1) over the base field k in terms of the Jacobians of curves over small extensions of k whenever possible. The decomposition of the Jacobian of a curve of genus 2 is also discussed in [27]. Similarly, when $A = \text{Jac}(Z)$ is the Jacobian of a curve of genus 3 that admits a degree-2 map $Z \rightarrow X$ to a genus-1 curve X , then the results in [35] furnish an explicit description of the complementary part B in the decomposition $A \sim \text{Jac}(X) \times B$ in terms of the Jacobian of a genus-2 curve Y .

This article aims to develop algorithms for the converse construction, that is, to produce an abelian variety A given factors B_i as in (0.1). When $A = \text{Jac}(Z)$ and $B_i = \text{Jac}(X_i)$, we also call the curve Z a gluing of the curves X_i .

Previous work. Gluing elliptic curves E_1 and E_2 to a genus-2 curve Z was first studied in the seminal article [17] by Frey and Kani, where explicit criteria for the existence of Z given E_1 and E_2 are given. In fact, [17] proves a more precise criterion, in that they also fix a degree n and realize Z

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as a degree- n cover of both E_1 and E_2 . Similarly, Howe–Leprévost–Poonen [26] use plane quartic curves with defining equation in the Ciani standard form

$$Z : ax^4 + by^4 + cz^4 + dx^2y^2 + ey^2z^2 + fz^2x^2 = 0 \quad (0.2)$$

to realize three given elliptic curves E_1, E_2, E_3 over k as factors of the Jacobian of Z .

Finally, [10] realizes the Jacobian of a given genus-2 curve over \mathbb{C} as part of the Jacobian of a smooth plane quartic, using modular methods. In the more recent preprint [11] a pencil of plane, bielliptic genus-three (and their unramified double coverings by canonical genus-five curves) is constructed such that the Prym variety of its general member is two-isogenous to the Jacobian of a generic genus-two curve. Like the results in [10] that are discussed below, those in [11] use that $k = \mathbb{C}$ and do not treat the case of an arithmetic base field.

Results in this paper. This paper considers the problem of gluing two curves of genus 1 and 2 to a curve of genus 3 along their 2-torsion over a given base field. More precisely, our main theorem, proved in Section 2, is as follows.

Main Theorem. *Let k be a field whose characteristic does not equal 2, and let X and Y be curves of genus 1 and 2 over k . Then Algorithm 2.52 returns all isomorphism classes of pairs (Z, μ) over k , where Z is a smooth plane quartic curve over k and where μ is a class in $k^*/(k^*)^2$ with the property that there exists a k -rational quotient map*

$$\text{Jac}(X) \times \text{Jac}(Y) \rightarrow \mu * \text{Jac}(Z) \quad (0.3)$$

*by a symplectic subgroup of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$. In particular, we have an isogeny $\text{Jac}(X) \times \text{Jac}(Y) \sim \mu * \text{Jac}(Z)$ over k . Here $\mu * \text{Jac}(Z)$ denotes the quadratic twist of $\text{Jac}(Z)$ with respect to the automorphism -1 and the quadratic extension $k(\sqrt{\mu})$ of k .*

The alternative Algorithm 3.88 in Section 3 also determines equations for curves Z gluing X and Y , but these may require a further base extension.

Our results function over any base field k of characteristic not equal to 2, not necessarily algebraically closed, which makes them relevant in the broader arithmetic-geometric context. Moreover, they allow one to specify both curves X and Y , whereas for the previous results [10] over the special base field $k = \mathbb{C}$ only Y could be specified. We mention here that there is also a short and simple construction of Z over a general base field k once only Y is specified: It is given in [22, §2.2] and involves the parametrization of a certain conic admitting a k -rational point. Similarly, the case where the glued curve Z is hyperelliptic will be dealt with in [23]. This article restricts its consideration to the more involved case where both X and Y are specified and where Z is smooth plane quartic curve.

Applications. Being able to work with decompositions (0.1) over arbitrary base fields, which our Main Theorem contributes to, is of arithmetic importance, for example when describing L -functions: If an abelian surface over \mathbb{Q} splits up to isogeny as $A \sim E_1 \times E_2$ over \mathbb{Q} , then we have $L(A, s) = L(E_1, s)L(E_2, s)$, and the modular properties of A can be reduced to those of E_1 and E_2 . This description extends to situations where the decomposition requires an extension of the base field. For a systematic exploration this topic for abelian surfaces, we refer to [6].

Another application is in the context of Sato–Tate groups, and more specifically around the recent classification [16] in genus 3. While many Sato–Tate groups in *loc. cit.* can be realized in a trivial way by abelian threefolds of the form $\text{Jac}(X) \times \text{Jac}(Y)$, it remains a challenge to see if the same can be achieved when considering Jacobians $\text{Jac}(Z)$ of curves Z of genus 3. The current work allows one to approach the latter problem by starting from suitable products $\text{Jac}(X) \times \text{Jac}(Y)$ and attempting to find a corresponding gluing as described in the Main Theorem (and in the next subsection).

Finally, over $k = \mathbb{C}$, the type of decomposition that we reconstruct is important in the context of certain integrable systems, see [15].

Outline. We now give a more precise description of our methods, as well as of some important intermediate results on the way to the Main Theorem. We will specify our curves X and Y of genus 1 and 2 by equations

$$X : y^2 = p_X(x) \tag{0.4}$$

and

$$Y : y^2 = p_Y(x) \tag{0.5}$$

over k . While general curves over k of genus 1 do not allow a defining equation (0.4), we may reduce to this case, since our constructions only involve the Jacobian of X , which can always be realized through such an equation.

For an isogeny (0.3) to exist, we need a criterion for the existence of a maximal isotropic subgroup G of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$ that is indecomposable (in the sense of not being a product of subgroups of $\text{Jac}(X)$ and $\text{Jac}(Y)$). This is furnished by Theorem 1.51, which is the following:

Subgroup Existence Criterion. *Let X and Y be as above. There exists an indecomposable maximal isotropic subgroup G of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$ that is defined over k if and only if*

- (i) p_Y admits a quadratic factor q_Y over k ;
- (ii) For the complementary factor $r_Y = p_Y/q_Y$ we have that the cubic resolvents $\varrho(p_X)$ and $\varrho(r_Y)$ have isomorphic splitting fields over k .

Note that this criterion allows one to work with simpler splitting fields than those defined by the factors p_X and r_Y themselves.

The existence of such a subgroup G does not always guarantee that the corresponding quotient $(\text{Jac}(X) \times \text{Jac}(Y))/G$ is the twist of the Jacobian of a plane quartic. Indeed, the polarization on said quotient may be decomposable, or give rise to a hyperelliptic curve. Generically, however, a quotient by a non-decomposable group G is isomorphic as a principally polarized abelian variety to a quadratic twist $\mu * \text{Jac}(Z)$ of a Jacobian of a plane quartic curve Z , as defined in the Main Theorem. Taking a twist is indispensable, as over an arithmetic base field most principally polarized abelian threefolds are not Jacobians — for a geometric description of this so-called Serre obstruction, see for example the main result in [4] quoted below as Theorem 1.57.

In Section 2, we proceed to find an expression for the curve Z in terms of the data in the Subgroup Existence Criterion whenever possible. The resulting curve Z will admit a homogeneous ternary quartic equation of the form

$$Z : G(x^2, y, z) = 0 \tag{0.6}$$

from which $\text{Jac}(X)$ can be recovered as the Jacobian of the quotient by the involution $(x, y, z) \mapsto (-x, y, z)$. Note that there is no direct map $Z \rightarrow Y$ in general, nor need it exist over the algebraic closure \bar{k} . In fact, Proposition 3.2 shows that this can only happen when Z is hyperelliptic, a case that we excluded from consideration. Therefore we cannot use constructions that only involve covers of curves.

Section 2 takes the following indirect route to constructing Z . We start by interpolating results over the complex numbers. When appropriately normalized, these yield formulae that can be verified a posteriori to remain valid over any field of characteristic not equal to 2. Note that evaluating these formulae, as is done in Algorithm 2.52 corresponding to the Main Theorem, once again only requires passing to the common splitting field of the aforementioned cubic resolvents, and is therefore feasible in practice also when the coefficients of the defining equations of X and Y are large. The formulae also yield the twisting scalar μ mentioned in the Main Theorem. Moreover, since they are obtained in a highly canonical fashion, applying them in concrete cases such as Example 2.68 yields small defining coefficients for Z without any further simplification being required.

An alternative and more geometric version of the Main Theorem is given in Section 3. It is essentially a geometric inversion of the results in the seminal paper [35] by Ritzenthaler and Romagny, and constructs Z as a double cover of X obtained by realizing X birationally as a hyperplane section of the Kummer variety of Y . This is the content of Theorem 3.14, which we state here as follows.

Geometric Main Theorem. *Let Z be a gluing of X and Y as in the Main Theorem. Let $\text{Kum}(Y)^t = \text{Jac}(Y)^t / \langle -1 \rangle \subset \mathbb{P}_k^3$ be the Kummer surface associated to $\text{Jac}(Y)^t$. Then over \bar{k} there exists a commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{i_Z} & \text{Jac}(Y)^t \\ p \downarrow & & \downarrow \pi \\ X & \xrightarrow{i_X} & \text{Kum}(Y)^t. \end{array} \quad (0.7)$$

where $p : Z \rightarrow X$ is a degree-2 cover, $\pi : \text{Jac}(Y)^t \rightarrow \text{Kum}(Y)^t$ is the quotient map, and where i_Z and i_X are rational maps such that $i_X(X) = H \cap \text{Kum}(Y)^t$ for a plane $H \subset \mathbb{P}_k^3$ that passes through two singular points of $\text{Kum}(Y)^t$.

Explicitly, an element of the function field of X whose square root gives rise to the double cover $Z \rightarrow X$ can be obtained by restricting a Kummer generator of the extension of function fields $k(\text{Kum}(Y)^t) \subset k(\text{Jac}(Y)^t)$ (which is described in [32]) to the hyperplane section X . Various tricks help make this calculation feasible in practice, especially over finite fields, and many interesting phenomena in this geometric Ansatz remain to be explained and generalized.

We give examples of the aforementioned constructions, both over \mathbb{Q} and over finite fields. Moreover, in Section 2.7 we use our results to obtain a Jacobian of a plane quartic curve over \mathbb{Q} whose quadratic twist with respect to the automorphism -1 and the extension $\mathbb{Q}(\sqrt{5})$ of \mathbb{Q} admits a rational 70-torsion point. A full implementation of the results in this article is openly available via a full MAGMA implementation and example suite at [24].

Notations and conventions. Throughout the article, k denotes a fixed base field, whose characteristic we suppose not to equal 2. Its absolute Galois group is denoted by Γ_k .

A curve over k is a separated and geometrically integral scheme of dimension 1 over k . Given an affine equation for a curve, we will identify it with the smooth projective curve that has the same function field. The Jacobian of a curve X is denoted by $\text{Jac}(X)$, and its principal polarization, which we consider as an algebraic equivalence class of line bundles on X , is denoted by λ_X .

1. CRITERIA FOR A GLUING TO EXIST

Let $n \geq 2$ be a positive integer, and let X (resp. Y) be a smooth curve of genus 1 (resp. 2) over the base field k , whose characteristic we suppose not to divide n . Let $\pi_X : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(X)$ and $\pi_Y : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Y)$ be the two canonical projections.

Definition 1.1. An (n,n) -gluing of the curves X and Y (over k) is a pair (Z, φ) , where Z is a smooth curve over k and where

$$\varphi : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Z) \quad (1.2)$$

is an isogeny with the property that $\varphi^*(\lambda_Z)$ is algebraically equivalent to the n -fold $n(\pi_X^*(\lambda_X) \otimes \pi_Y^*(\lambda_Y))$ of the product polarization on $\text{Jac}(X) \times \text{Jac}(Y)$.

Let

$$T = (\text{Jac}(X) \times \text{Jac}(Y))[n] = \text{Jac}(X)[n] \times \text{Jac}(Y)[n]. \quad (1.3)$$

Let $V = T(\bar{k})$, and consider a maximal isotropic subgroup G of V . Then over \bar{k} we can form the fppf quotient

$$Q = (\text{Jac}(X) \times \text{Jac}(Y))/G. \quad (1.4)$$

Let

$$\pi_Q : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow Q \quad (1.5)$$

be the quotient morphism. By [38, Proposition 11.25], there exists a unique principal polarization λ_Q on Q whose pullback under π_Q is algebraically equivalent to $n(\pi_X^*(\lambda_X) \otimes \pi_Y^*(\lambda_Y))$. Since we have imposed that $\text{char}(k)$ does not divide n , so that T is étale, and the quotient morphism Q is defined over k if and only if the subgroup G is, we obtain the following from the arithmetic version of Torelli's theorem.

Lemma 1.6. *Giving a gluing (Z, φ) of X and Y over k is the same as giving a maximal isotropic subgroup G of V with the following properties.*

- (i) G is stable under the action of the absolute Galois group Γ_k .
- (ii) Moreover, the principally polarized quotient (Q, λ_Q) constructed from G is k -isomorphic to the Jacobian of a smooth curve over k .

Remark 1.7. Condition (ii) in Lemma 1.6 implies Condition (i), but is strictly stronger than it. For example, it implies that the group G cannot be decomposable, that is, a product $G_X \times G_Y$ of maximal isotropic subgroups of $\text{Jac}(X)[n]$ and $\text{Jac}(Y)[n]$. Indeed, in this case we have

$$Q \cong \text{Jac}(X)/G_X \times \text{Jac}(Y)/G_Y \quad (1.8)$$

and λ_Q is the corresponding product polarization. This precludes the existence of an isomorphism $(\text{Jac}(Z), \lambda_Z) \cong (Q, \lambda_Q)$, since principal polarizations of Jacobians are indecomposable.

The demand in Condition (ii) that (Q, λ_Q) be a Jacobian over the ground field k itself leads to additional arithmetic subtleties, which will be discussed in Section 1.4. We will theretofore first concentrate on developing criteria that focus on the weaker Condition (i) only.

1.1. Structure of maximally isotropic subgroups. In the situation of Definition 1.1, let $n = p$ be a prime number. This simplifies the description of the maximal isotropic subgroups G of $V = T(\bar{k})$. Let $V_X = \text{Jac}(X)[p](\bar{k})$ and $V_Y = \text{Jac}(Y)[p](\bar{k})$, so that $V = V_X \times V_Y$. Moreover, let

$$E = E_X \times E_Y : V \times V \rightarrow \mathbb{F}_p \quad (1.9)$$

be the product of the Weil pairings E_X and E_Y on V_X and V_Y . Finally, let $\pi_X : G \rightarrow V_X$ and $\pi_Y : G \rightarrow V_Y$ be the canonical projections.

Proposition 1.10. *If G is indecomposable, then π_X is surjective and we have*

$$\dim(\ker(\pi_X)) = \dim(G \cap (0 \times V_Y)) = 1. \quad (1.11)$$

Proof. We have $\dim(G) = 3$. If $\dim(\text{im}(\pi_X)) = 0$, then $\ker(\pi_X)$ can be identified with an isotropic subgroup of $0 \times V_Y$ of dimension 3. No such subgroups exist since V_Y has dimension 4. Therefore $\dim(\text{im}(\pi_X)) = 1$, which implies that $\text{im}(\pi_X)$ is a symplectic subgroup of V_X , so that for all (x_1, y_1) and (x_2, y_2) in $G \subset V_X \times V_Y$ we have

$$0 = E(((x_1, y_1), (x_2, y_2))) = E_X((x_1, x_2)) + E_Y((y_1, y_2)) = E_Y((y_1, y_2)). \quad (1.12)$$

This implies that $\text{im}(\pi_Y)$ is an isotropic subgroup of V_Y . Since $G \subset \text{im}(\pi_X) \times \text{im}(\pi_Y)$ this forces $\dim(\text{im}(\pi_Y)) = 2$ and $G = \text{im}(\pi_X) \times \text{im}(\pi_Y)$. This is a contradiction with the indecomposability of G . The second statement of the proposition then follows from the dimension formula. \square

Now fix an indecomposable maximal isotropic subgroup $G \subset V_X \times V_Y$. Let $H \subset V_Y$ be the 1-dimensional \mathbb{F}_p -subspace defined by

$$H = \pi_Y(\ker(\pi_X)) = \pi_Y(G \cap (0 \times V_Y)) \subset V_Y. \quad (1.13)$$

In other words, H is the subgroup of V_Y generated by the second components of the vectors of the form $(0, w)$ in G . The orthogonal complement $H^\perp \subset V_Y$ is of dimension 3, so that we have

$$\dim(H^\perp/H) = 2. \quad (1.14)$$

The symplectic pairing E_Y on V_Y induces one on H^\perp/H , which we will denote by E_\perp . There is a multivalued map

$$V_X \rightarrow H^\perp/H \quad (1.15)$$

that sends $x \in V_X$ to $\pi_Y(\pi_X^{-1}(x))$. Since $H \times 0 \subset G$ and G is isotropic, the elements of $\pi_Y(\pi_X^{-1}(x))$ are in H^\perp . The map (1.15) factors to a single-valued linear map

$$\ell : V_X \rightarrow H^\perp/H. \quad (1.16)$$

Now for all (x_1, y_1) and (x_2, y_2) in G we have

$$0 = E((x_1, y_1), (x_2, y_2)) = E_X((x_1, x_2)) + E_Y((y_1, y_2)), \quad (1.17)$$

By construction of ℓ we have $y_i + H = \ell(x_i) + H$. Therefore the map ℓ is antisymplectic. In particular, ℓ is an isomorphism.

Thus an indecomposable maximal isotropic subgroup G gives rise to a subgroup $H \subset V_Y$ and an anti-symplectic linear isomorphism $\ell_G : V_X \rightarrow H^\perp/H$. As is shown in more detail in [21], there is a converse to this result:

Proposition 1.18. *Let (H, ℓ) be a pair with $H \subset V_Y$ of dimension 1 and with $\ell : V_X \rightarrow H^\perp/H$ an anti-symplectic isomorphism. Define*

$$G = \{(x, y) \in V_X \times V_Y : \ell(x) = y + H\}. \quad (1.19)$$

Then $G \subset V_X \times V_Y$ is indecomposable and maximal isotropic.

This construction of G_ℓ from (H, ℓ) is inverse to that of (H, ℓ) from G above and yields a bijective correspondence between indecomposable maximal isotropic subgroups $G \subset V_X \times V_Y$ on the one hand and the pairs (H, ℓ) under consideration on the other.

Proof. The first part follows by the same methods as above. Note in particular that G is indecomposable because its intersection with $0 \times V_Y$ is of dimension 1. The remainder of the statement follows by direct verification. \square

Corollary 1.20. *There exist exactly $p(p+1)(p^4-1)$ indecomposable maximal isotropic subgroups of $V_X \times V_Y$.*

Proof. By Proposition 1.18, giving such an indecomposable maximal isotropic subgroup is the same as giving a pair (H, ℓ) . Since V_Y has dimension 4, there are $(p^4-1)/(p-1)$ possible ways to choose H . Given H , there are $\#\mathrm{Sp}(2, \mathbb{F}_p) = \#\mathrm{SL}_2(\mathbb{F}_p) = p(p^2-1)$ choices for the anti-symplectic isomorphism ℓ . Taking the product yields the cardinality in the statement of the corollary. \square

Remark 1.21. The total number of maximal isotropic subgroups of $V_X \times V_Y$ (including the indecomposable ones) equals $(p^3+1)(p^2+1)(p+1)$.

From now on, we restrict to the case $p=2$, in which case we simply call a $(2, 2)$ -gluing a **gluing**. The number of indecomposable maximal isotropic subgroups of $V_X \times V_Y$ then equals 135, of which 90 are indecomposable in our sense. Note that [26, Lemma 13] classifies indecomposable subgroups in another context than ours, namely when gluing three genus-1 curves together; $V_X \times V_Y$ has 54 indecomposable subgroups in this alternative sense.

1.2. Interpretation in terms of roots. Having taken $p = 2$, the symplectic vector spaces V_X and V_Y admit the following concrete descriptions. Choose quadratic defining equations

$$X : y^2 = p_X(x) \tag{1.22}$$

and

$$Y : y^2 = p_Y(x) \tag{1.23}$$

over \bar{k} , as one may since $\text{char}(k) \neq 2$. For now, suppose that p_X (resp. p_Y) is of degree 4 (resp. 6). Let

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \bar{k} \tag{1.24}$$

be the roots of p_X , and let

$$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \bar{k} \tag{1.25}$$

be the roots of p_Y . Consider the corresponding sets

$$\mathcal{P} = \left\{ P_i = (\alpha_i, 0) \in X(\bar{k}) : i \in \{1, \dots, 4\} \right\}, \quad \mathcal{Q} = \left\{ Q_j = (\beta_j, 0) \in Y(\bar{k}) : j \in \{1, \dots, 6\} \right\}. \tag{1.26}$$

Remark 1.27. If either one of the degrees of p_X and p_Y is odd, then we can formally consider ∞ as an element of \mathcal{P} or \mathcal{Q} (or both). To simplify the exposition, we ignore these cases; of course our final results still take them into account.

Given a set T of even cardinality, we can define a symplectic \mathbb{F}_2 -vector space $\mathcal{G}(T)$ as follows: The elements of $\mathcal{G}(T)$ are the subsets S of T of even cardinality up to the equivalence $S \sim S^c$. The symmetric difference operation

$$(S_1, S_2) \mapsto S_1 \oplus S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2). \tag{1.28}$$

descends to a group structure on $\mathcal{G}(T)$, for which the empty subgroup corresponds to the identity element. Finally, we equip $\mathcal{G}(T)$ with the symplectic pairing

$$\begin{aligned} \mathcal{G}(T) \times \mathcal{G}(T) &\rightarrow \mathbb{F}_2 \\ (S_1, S_2) &\mapsto \#(S_1 \cap S_2) \bmod 2. \end{aligned} \tag{1.29}$$

Now [33] shows the following.

Proposition 1.30. *The symplectic \mathbb{F}_2 -vector space V_X (resp. V_Y) can be identified with the $\mathcal{G}(\mathcal{P})$ (resp. $\mathcal{G}(\mathcal{Q})$). To the equivalence class \bar{S} of a subgroup $S = \{P_1, P_2\}$ of cardinality 2 there corresponds the 2-torsion point $[P_1] - [P_2]$.*

Remark 1.31. For the genus-2 curve Y , the subsets of \mathcal{Q} of cardinality 2 are in bijective correspondence with the non-zero elements of V_Y . Indeed, the subsets S of \mathcal{P} of even cardinality that do not give rise to $0 \in V_Y$ are of cardinality 2 or 4, so that exactly one of S and S^c is of cardinality 2.

By contrast, for the genus-1 curve X the non-zero elements of V_X are no longer in bijective correspondence with the subsets S of \mathcal{P} cardinality 2: this needs the identification of a set S with its complement S^c .

Consider a pair (H, ℓ) as in Proposition 1.18. In terms of \mathcal{P} and \mathcal{Q} , giving H is nothing but giving a subset \mathcal{T} of \mathcal{Q} of cardinality 2. Let $\mathcal{R} = \mathcal{Q} \setminus \mathcal{T}$ be the complement of \mathcal{T} .

Proposition 1.32. *The inclusion $\iota : \mathcal{R} \hookrightarrow \mathcal{Q}$ induces a canonical isomorphism of symplectic \mathbb{F}_2 -vector spaces*

$$i_{\mathcal{R}} : \mathcal{G}(\mathcal{R}) \rightarrow H^\perp / H. \tag{1.33}$$

Proof. Both vector spaces involved are of dimension 2. The injection ι gives rise to a well-defined map on equivalence classes \bar{S} since for $S \subset \mathcal{R}$ we have

$$\iota(S^c) = \iota(\mathcal{R} \setminus S) = \mathcal{Q} \setminus (\mathcal{T} \cup S) \sim \mathcal{T} \cup S. \quad (1.34)$$

Taking the symmetric difference with the non-trivial element \mathcal{T} in H , we obtain the class

$$(\mathcal{T} \cup S) \oplus \mathcal{T} = S = \iota(S). \quad (1.35)$$

This shows that the images of $\iota(S^c)$ and $\iota(S)$ in H^\perp/H indeed coincide.

The map i_R is linear (and hence symplectic) since

$$\iota(S_1 \oplus S_2) = \iota((S_1 \cup S_2) \setminus (S_1 \cap S_2)) = (S_1 \cup S_2) \setminus (S_1 \cap S_2) = \iota(S_1) \oplus \iota(S_2). \quad (1.36)$$

Finally, i_R is injective since the image of the equivalence class of a subset $S \subset R$ of cardinality 2 remains an equivalence class of a subset of cardinality 2 and is therefore non-trivial. We conclude that i_R is indeed a symplectic isomorphism. \square

Combining the above results, we get the following.

Corollary 1.37. *Giving an indecomposable maximal isotropic subgroup $G \subset V_X \times V_Y$ is the same as giving a subset \mathcal{T} of \mathcal{Q} of cardinality 2 along with a symplectic isomorphism*

$$\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R}), \quad (1.38)$$

where $\mathcal{R} = \mathcal{Q} \setminus \mathcal{T}$.

1.3. Rationality criteria. Having heretofore worked over the algebraic closure \bar{k} , we now consider criteria for Condition (i) in Lemma 1.6. This requires the application of Galois theory to detect when a maximal isotropic subgroup G is defined over the base field k .

We start by noting that we may assume X and Y to admit defining equations (1.22) and (1.23) over k . For Y this follows from the fact that every genus-2 curve over k admits such an equation, whereas for X we may make this assumption because only the Jacobian $\text{Jac}(X)$ intervenes in our constructions, and this Jacobian is an elliptic curve, which therefore admits an equation (1.22) since k is not of characteristic 2.

We proceed to give concrete criteria for the Galois stability in Part (i) of Lemma 1.6 in terms of the equivalent interpretation of the indecomposable maximal isotropic subgroup $G \subset V_X \times V_Y$ that we developed in the previous section.

Proposition 1.39. *Let (H, ℓ) be a pair with $H \subset V_Y$ of dimension 1 and with $\ell : V_X \rightarrow H^\perp/H$ an anti-symplectic isomorphism, and let G be the corresponding indecomposable subgroup of V , as described in Proposition 1.18. Then G is Galois stable if and only if:*

- (i) $H \subset V_Y$ is a Galois stable subspace of dimension 1;
- (ii) The map $\ell : V_X \rightarrow H^\perp/H$ is Galois equivariant.

Proof. If G is stable under the action of Γ_k , then since the same is true for V_Y , the intersection $H = G \cap (0 \times V_Y)$ is Galois stable. It has dimension 1 by Proposition 1.10, showing that (i) is necessary. The same holds for (ii). Indeed, if G is Galois stable, then if $(x, y) \in G$, the same holds for $(\sigma(x), \sigma(y))$. This in turn means that $\ell(\sigma(x)) = \sigma(y) + H$ in H^\perp/H , so indeed ℓ is Galois equivariant, as

$$\sigma(\ell(x)) = \sigma(y) + H = \ell(\sigma(x)). \quad (1.40)$$

Now suppose conversely that (H, ℓ) fulfills the conditions of the proposition. Let (x, y) be an element of the corresponding group G . We have $\ell(x) = y + H$. Since ℓ is Galois equivariant, we have

$$\ell(\sigma(x)) = \sigma(\ell(x)) = \sigma(y) + \sigma(H) = \sigma(y) + H, \quad (1.41)$$

which implies $(\sigma(x), \sigma(y)) \in G$. Therefore G is Galois stable. \square

In terms of our chosen defining equations for $X : y^2 = p_X$ and $Y : y^2 = p_Y$, the Galois stable subgroup H yields a quadratic factor q_Y of p_Y over k , which corresponds to the roots in the set \mathcal{T} . Let $r_Y = p_Y/q_Y$ be the complementary factor corresponding to the roots in $\mathcal{R} = \mathcal{Q} - \mathcal{T}$.

Recall that given a quartic polynomial

$$p = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in k[x], \quad (1.42)$$

its cubic resolvent $\varrho(p)$ is defined by

$$\varrho(p) = x^3 - a_2x^2 + (a_1a_3 + 4a_0)x + (4a_0a_2 - a_1^2 - a_0a_3^2) \in k[x] \quad (1.43)$$

For simplicity of exposition, we define the cubic resolvent of a general quartic polynomial as the cubic resolvent of the polynomial obtained by dividing it by its leading coefficient. The cubic resolvent $\varrho(p)$ is known to be separable if p is, and if $\alpha_1, \dots, \alpha_4$ are the roots of p , then the distinct roots of $\varrho(p)$ are given by

$$\gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3. \quad (1.44)$$

While a change of numbering of the roots $\alpha_1, \dots, \alpha_4$ permutes the expressions $\gamma_1, \dots, \gamma_3$, the set $S_p = \{\gamma_1, \dots, \gamma_3\}$ does remain invariant. The Galois action on S_p factors through the quotient $\text{Gal}(p)$ of Γ_k . Under the embedding $\text{Gal}(p) \hookrightarrow \text{Sym}(4)$ induced by the given numbering $\{\alpha_1, \dots, \alpha_4\}$, we have

$$\begin{aligned} \text{Stab}_{\text{Gal}(p)}(\gamma_1) &= \langle (1\ 3\ 2\ 4), (1\ 2) \rangle \cap \text{Gal}(p), \\ \text{Stab}_{\text{Gal}(p)}(\gamma_2) &= \langle (1\ 2\ 3\ 4), (1\ 3) \rangle \cap \text{Gal}(p), \\ \text{Stab}_{\text{Gal}(p)}(\gamma_3) &= \langle (1\ 2\ 4\ 3), (1\ 4) \rangle \cap \text{Gal}(p). \end{aligned} \quad (1.45)$$

The intersection of these stabilizers of γ_i corresponds to the subgroup $V_4 \cap \text{Gal}(p)$ of $\text{Gal}(p)$. Because of its normality, this group is independent of the chosen numbering of the roots of p . We see that the splitting field of $\varrho(p)$ is the number field corresponding to the subgroup $V_4 \cap \text{Gal}(p) \subset \text{Gal}(p)$ under the Galois correspondence for $\text{Gal}(p)$. A more precise consideration enables us to prove the following:

Proposition 1.46. *There exists a Galois equivariant isomorphism $\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$ if and only if the splitting fields of $\varrho(p_X)$ and $\varrho(r_Y)$ are isomorphic.*

Proof. Since the trivial element of $\mathcal{G}(\mathcal{P})$ is fixed by Γ_k , its Galois structure is determined by the action on the three non-trivial elements of $\mathcal{G}(\mathcal{P}) \setminus \{0\}$. We have the following isomorphism of Γ_k -sets:

$$\begin{aligned} \mathcal{G}(\mathcal{P}) \setminus \{0\} &\rightarrow S_{p_X} \\ \overline{\{P_1, P_2\}} &= \overline{\{P_3, P_4\}} \mapsto \gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \\ \overline{\{P_1, P_3\}} &= \overline{\{P_2, P_4\}} \mapsto \gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \\ \overline{\{P_1, P_4\}} &= \overline{\{P_2, P_3\}} \mapsto \gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3. \end{aligned} \quad (1.47)$$

Indeed, the action factors through the Galois group of the splitting field of p_X , and the induced action of $\text{Sym}(4)$ on the indices gives rise to an identical action on elements on both sides of (1.47).

It therefore remains to be analyzed when the Γ_k -sets S_{p_X} and S_{r_Y} are isomorphic. Certainly an isomorphism of these sets induces an isomorphism of the splitting fields of p_X and r_Y . Conversely, if these splitting fields are isomorphic, then ϱ_X and ϱ_Y have a common kernel N . These representations are therefore conjugate by the following lemma, which uses that the polynomials p_X and r_Y are cubic in an essential way. \square

Lemma 1.48. *Let Γ be a group, and let*

$$\varrho_1, \varrho_2 : \Gamma \rightarrow \text{Sym}(3) \quad (1.49)$$

be two representations. Then ρ_1 and ρ_2 are $\text{Sym}(3)$ -conjugate if and only if their kernels coincide and their images in $\text{Sym}(3)$ are isomorphic.

Proof. This is a direct consequence of the fact that two subgroups of $\text{Sym}(3)$ are conjugate if and only if they are isomorphic. \square

Remark 1.50. It is not much more difficult to calculate the number of Galois equivariant isomorphisms $\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$. Indeed, if the given Γ_k -sets are conjugate, then the number of isomorphisms is nothing but their common number of automorphisms, which in turn is the number of elements of the centralizer of either of their images in $\text{Sym}(3)$.

Theorem 1.51. *Let X (resp. Y) be a curve of genus 1 (resp. 2) admitting a defining equation $X : y^2 = p_X$ (resp. $Y : y^2 = p_Y$). There exists a Galois stable indecomposable maximal isotropic subgroup $G \subset V_X \times V_Y$ if and only if*

- (i) p_Y admits a quadratic factor q_Y over k ;
- (ii) For the complementary factor $r_Y = p_Y/q_Y$ we have that the cubic resolvents $\varrho(p_X)$ and $\varrho(r_Y)$ have isomorphic splitting fields over k .

Proof. This follows by combining Proposition 1.39(i) and Proposition 1.46. \square

Remark 1.52. As mentioned at the start of Section 1.2, some changes take place when either p_X or p_Y is of odd degree. If p_Y has degree 5, then we should also consider the case where p_Y has a linear factor over k in Part (i) of Theorem 1.51. Similarly, if p_X or r_Y is of odd degree, then we should use this polynomial directly in Part (ii) instead of taking a Galois resolvent.

1.4. Intervening twists. The previous section has given a concrete characterization of Part (i) of Lemma 1.6. For Part (ii), we restrict ourselves in this article to the case where the quotient $(\text{Jac}(X) \times \text{Jac}(Y))/G$ is a Jacobian over \bar{k} . When $\bar{k} \subset \mathbb{C}$, it is possible to characterize the case when this occurs by numerical complex-analytic methods, which are further discussed in Section 2: namely, none of the even theta-null values of the complex torus corresponding to $(\text{Jac}(X) \times \text{Jac}(Y))/G$ should vanish.

Remark 1.53. We note in passing and without detail that it is not difficult to characterize when there exists a hyperelliptic gluing (Z, φ) over \bar{k} . This is the case if and only if one of the cross ratios of the roots of the polynomial p_X that defines X coincides with one of the cross ratios of four of the roots of the polynomial p_Y that defines Y , as one observes by noting that the Prym variety of the obvious morphism from

$$Z : y^2 = x^8 + a_3x^6 + a_2x^4 + a_1x^2 + a_0 \tag{1.54}$$

to

$$X : y^2 = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \tag{1.55}$$

is given by

$$Y : y^2 = x(x^4 + a_3x^3 + a_2x^2 + a_1x + a_0). \tag{1.56}$$

However, issues of rationality remain to be explored thoroughly. Note in particular that it is possible for the glued curve Z to be hyperelliptic over \bar{k} but not over k , which happens when Z is a degree-2 cover of a non-trivial conic. The upcoming work [23] will explore these themes in detail.

For non-algebraically closed fields, there is an additional complication: It is possible that (Z, λ_Z) is a Jacobian over \bar{k} , but not over k . More precisely, we have the following by [4]:

Theorem 1.57. *Let (Q, λ_Q) be a principally polarized abelian threefold over k that is not a product of abelian varieties of smaller dimension over \bar{k} . Then there exists a curve Z over k and a field extension $\ell|k$ with $[\ell : k] \leq 2$ such that (Q, λ_Q) is isomorphic to the quadratic twist of $(\text{Jac}(Z), \lambda_Z)$*

with respect to the automorphism -1 and the extension $\ell|k$. Moreover, if Z is hyperelliptic, then $\ell = k$.

Given a curve Z over k and a scalar $\mu \in k^*$, we more concisely define quadratic twist of $(\text{Jac}(Z), \lambda_Z)$ with respect to -1 by μ to be the quadratic twist of $(\text{Jac}(Z), \lambda_Z)$ with respect to the automorphism -1 and the extension $k(\sqrt{\mu})|k$. Note that this twist only depends on the class of μ in $k^*/(k^*)^2$.

Definition 1.58. Let $Q = (\text{Jac}(X) \times \text{Jac}(Y))/G$ and Z be as in the preceding theorem, and let $\mu \in k^*$. If (Q, λ_Q) is isomorphic to the quadratic twist of $(\text{Jac}(Z), \lambda_Z)$ with respect to -1 by μ , then we call the curve Z a twisted gluing of X and Y (with twisting scalar μ).

Given generic X and Y , the twisting scalar needed to obtain the quotient Q from $\text{Jac}(Z)$ is usually non-trivial. In other words, over a general ground field k a twisted gluing is more likely to exist than an actual gluing (and even the existence of a twisted gluing requires the fairly restrictive hypotheses of Theorem 1.51). In the plane quartic case that we will consider in the following section, we will calculate both Z and the twisting scalar μ explicitly in terms of the polynomials p_X and p_Y that define the curves X and Y and the cubic resolvents involved in Theorem 1.51.

2. GLUING VIA INTERPOLATION

2.1. Numerical algorithms over \mathbb{C} . Consider the base field $k = \mathbb{C}$. In this section, we consider gluings from an analytic point of view. As in Section 1.2, we choose defining equations

$$X : y^2 = p_X(x) \tag{2.1}$$

and

$$Y : y^2 = p_Y(x). \tag{2.2}$$

Consider the sets of roots $\mathcal{P} = \{\alpha_1, \dots, \alpha_4\}$ of p_X and $\mathcal{Q} = \{\beta_1, \dots, \beta_6\}$ of p_Y . Let $\mathcal{T} = \{\beta_5, \beta_6\}$ and $\mathcal{R} = \mathcal{Q} - \mathcal{T}$. Via the correspondence in Corollary 1.37, we consider the maximal isotropic subgroup G defined by the pair (\mathcal{T}, ℓ) , where $\mathcal{T} = \{\beta_5, \beta_6\}$ and where ℓ is defined as

$$\begin{aligned} \ell : \mathcal{G}(\mathcal{P}) &\rightarrow \mathcal{G}(\mathcal{R}) \\ \overline{\{\alpha_1, \alpha_i\}} &\mapsto \overline{\{\beta_1, \beta_i\}}. \end{aligned} \tag{2.3}$$

for $i \in \{2, 3, 4\}$. In other words, we fix a root pairing determined by our choice of ordering of the roots of p_X and p_Y . We intend to find a corresponding genus-3 curve defined by a homogeneous ternary quartic equation

$$Z : F_Z(x, y, z) = 0 \subset \mathbb{P}^2 \tag{2.4}$$

provided that such an equation exists.

Definition 2.5. Let X be a curve over \mathbb{C} , and let $\mathcal{B} = \{\omega_1, \dots, \omega_g\}$ be a basis of the \mathbb{C} -vector space of global differentials $H^0(X, \omega_X)$. The period lattice $\Lambda_{X, \mathcal{B}}$ of X with respect to \mathcal{B} is the lattice in \mathbb{C}^g defined by

$$\Lambda_{X, \mathcal{B}} = \left\{ \left(\int_{\gamma} \omega_i \right)_{i=1, \dots, g} : \gamma \in H_1(X, \mathbb{Z}) \right\}. \tag{2.6}$$

It carries a natural symplectic pairing $E_{\Lambda_{X, \mathcal{B}}}$ coming from the intersection product. We call a pair $(\Lambda_{X, \mathcal{B}}, E_{\Lambda_{X, \mathcal{B}}})$ thus obtained from some curve X of genus g and some basis \mathcal{B} of $H^0(X, \omega_X)$ a period lattice with a distinguished symplectic pairing in \mathbb{C}^g . When its presence is clear from the context, we remove mention of $E_{\Lambda_{X, \mathcal{B}}}$.

A choice (2.1) of defining equation picks out a distinguished basis of $H^0(X, \omega_X)$, namely

$$\mathcal{B}_{p_X} = \{dx/y\} \quad (2.7)$$

Similarly, the defining equations (2.2) and (2.4) pick out the bases

$$\mathcal{B}_{p_Y} = \{xdx/y, dx/y\} \quad (2.8)$$

and

$$\mathcal{B}_{F_Z} = \{xdx/(\partial f_Z/\partial y), ydx/(\partial f_Z/\partial y), dx/(\partial f_Z/\partial y)\}, \quad (2.9)$$

where $f_Z(x, y) = F_Z(x, y, 1)$. For genus-2 curves, this yields a map

$$\left\{ \begin{array}{l} \text{Defining polynomials } p_Y \text{ in (2.2)} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Period lattices with a distinguished} \\ \text{symplectic pairing in } \mathbb{C}^2 \end{array} \right\} \quad (2.10)$$

$$p_Y \mapsto \Lambda_{Y, \mathcal{B}_{p_Y}}.$$

Similarly, for smooth plane quartic curves we obtain a map

$$\left\{ \begin{array}{l} \text{Defining equations } F_Z \text{ in (2.4)} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Period lattices with a distinguished} \\ \text{symplectic pairing in } \mathbb{C}^3 \end{array} \right\} \quad (2.11)$$

$$F_Z \mapsto \Lambda_{Z, \mathcal{B}_{F_Z}}.$$

Proposition 2.12. *The map (2.10) is bijection, and the map (2.11) yields a bijection after quotienting out the identification $F_Z \sim -F_Z$.*

Proof. We first prove surjectivity of (2.10). By definition, given a period lattice with symplectic pairing Λ , there exists a pair (Y, \mathcal{B}) that gives rise to it. We may assume that Y is defined by a polynomial p_Y as in (2.8). At this point, the sole remaining problem is that the basis \mathcal{B} need not be the distinguished basis \mathcal{B}_{p_Y} associated to p_Y .

Given a matrix $S \in \text{GL}_2(\mathbb{C})$ and a defining polynomial p as in (2.8), we can transform p by the fractional linear transformation in x induced by S^{-1} to obtain a new polynomial $p \cdot S^{-1}$ as well as a canonical isomorphism from the curve defined by p to the curve defined by $p \cdot S^{-1}$. Explicitly, when $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this isomorphism is given by

$$(x, y) \mapsto \left(\frac{ax + b}{cx + d}, \frac{y}{cx + d} \right). \quad (2.13)$$

Under the map on differentials induced by this isomorphism, the distinguished basis $\mathcal{B}_{p \cdot S^{-1}}$ pulls back to a scalar multiple of the (in general no longer distinguished) basis $S \cdot \mathcal{B}_p$, where the latter action of S is defined by treating \mathcal{B}_p as a vector.

Since \mathcal{B} and \mathcal{B}_{p_Y} are bases of global differentials for the same curve Y , we have $T \cdot \mathcal{B} = \mathcal{B}_{p_Y}$ for some $T \in \text{GL}_2(\mathbb{C})$. The argument above shows that under the canonical isomorphism of Y with the curve defined by $p_Y \cdot T$, the basis \mathcal{B} is a scalar multiple of the distinguished basis. Replacing p_Y by $p_Y \cdot T$ if necessary, we may therefore assume that (Y, \mathcal{B}_{p_Y}) gives rise to a scalar multiple of Λ . As scaling the polynomial p_Y induces a scaling of the basis of differentials \mathcal{B}_Y , we can therefore indeed find a polynomial giving rise to the specified period lattice with symplectic pairing Λ under (2.10).

For injectivity we apply Torelli's theorem: Because the symplectic pairing that is (implicitly) present on the right-hand side of (2.10) is fixed, we know that if p_1 and p_2 on the left-hand side give rise to the same period matrix with symplectic pairing, there in fact exists an isomorphism between the corresponding hyperelliptic curves. Following the argument for surjectivity shows that there even exists an isomorphism whose corresponding fractional linear transformation in x is trivial. The corresponding map in y is necessarily a scaling by $\lambda \in \mathbb{C}$, which only fixes a given period lattice

when $\lambda \in \{\pm 1\}$. Since transforming by the hyperelliptic involution does not affect the defining equations of hyperelliptic curves, we conclude that $p_1 = p_2$.

For plane quartics, the same argument can be run for surjectivity, except that this time $T \in \mathrm{GL}_3(\mathbb{C})$ acts via fractional linear transformation on the projective coordinates x, y, z . For injectivity, given two defining equations $F_1 = 0$ and $F_2 = 0$ giving rise to the same period matrix with symplectic pairing Λ , we can again conclude the existence of an isomorphism between the curves defined by $F_1 = 0$ and $F_2 = 0$ whose tangent representation is a scalar. We conclude that $F_2 = \mu F_1$ for a scalar μ . As multiplying a ternary quartic F by μ scales the lattice Λ by μ^{-1} , we conclude that there remains the ambiguity of scaling F by $\mu = -1$. Conversely, no other such scalings fix the lattice $\Lambda \subset \mathbb{C}^3$, so that we obtain a bijection after quotienting out the identification in the statement of the proposition. \square

Remark 2.14. An explicit algorithm to recover the polynomial p_Y in (2.10) that corresponds to a given period lattice with a symplectic pairing is given in [20].

Remark 2.15. The statement in Proposition 2.12 does not hold for defining polynomials of genus-1 curves X . Indeed, the curves

$$X_1 : y^2 = x(x-1)(x-2) \quad (2.16)$$

and

$$X_2 : y^2 = (x+1)x(x-1) \quad (2.17)$$

define the same period lattice, as the isomorphism $(x, y) \mapsto (x-1, y)$ from X_1 to X_2 preserves the regular differential dx/y . Yet the defining polynomials of X_1 and X_2 are not equal.

Let Λ_X and Λ_Y be the period lattices resulting from the choices of basis (2.7) and (2.8). These matrices can be calculated especially fast by using new algorithms by Molin–Neurohr [31]. This functionality also includes the calculation of the Abel–Jacobi map, which means that we can construct elements of the group G , as follows.

Given a hyperelliptic curve X with Weierstrass points P_1 and P_2 , the corresponding element of $H^0(X, \omega_X)^*/H_1(X, \mathbb{Z})$ under the Abel–Jacobi map is

$$\omega \mapsto \int_{P_1}^{P_2} \omega, \quad (2.18)$$

where the integral can be taken along any path between P_1 and P_2 . Now the period lattice corresponding to $\mathrm{Jac}(X) \times \mathrm{Jac}(Y)$ is

$$\Lambda = \Lambda_X \times \Lambda_Y \subset \mathbb{C}^1 \times \mathbb{C}^2 = \mathbb{C}^3. \quad (2.19)$$

Because of the construction in Proposition 1.18 and the above remark, a basis for $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ is given by the elements $\{v_1, v_2, v_3\}$ of

$$\left\{ \left(0, \int_{\beta_5}^{\beta_6} x dx/y, \int_{\beta_5}^{\beta_6} dx/y \right), \left(\int_{\alpha_1}^{\alpha_2} dx/y, \int_{\beta_1}^{\beta_2} x dx/y, \int_{\beta_1}^{\beta_2} dx/y \right), \left(\int_{\alpha_1}^{\alpha_3} dx/y, \int_{\beta_1}^{\beta_3} x dx/y, \int_{\beta_1}^{\beta_3} dx/y \right) \right\} \quad (2.20)$$

of \mathbb{C}^3/Λ . Choosing a basis $\{e_1, \dots, e_6\}$ of the \mathbb{Z} -module Λ , we can use numerical calculation to find $q_{i,j} \in (1/2)\mathbb{Z}$ such that

$$v_i = q_{i,1}e_1 + \dots + q_{i,6}e_6. \quad (2.21)$$

Using linear algebra over \mathbb{Z} , we can find a basis for the lattice Λ_Z obtained by adjoining the elements of G . Moreover, the principal polarization on Λ (which is returned by the algorithms in [31]) extends to a polarization on Λ_Z , which by construction of G is a 2-fold of a principal polarization E .

Our task is to find a plane quartic equation (2.4) corresponding to (Λ_Z, E) , if it exists. This is accomplished by the following algorithm, whose implementation builds on the essential algorithms in [34].

Algorithm 2.22. This algorithm (numerically) reconstructs a ternary quartic from a period lattice.

Input: A period lattice Λ with a principal polarization E .

Output: A ternary plane quartic F whose corresponding period lattice with respect to (2.9) equals Λ , if it exists.

Steps:

- (i) Choose a matrix $P = (P_1 \ P_2) \in M_{3,6}(\mathbb{C})$ with respect to a symplectic basis for E , and let $\tau = P_2^{-1}P_1$ be a corresponding small period matrix.
- (ii) As in [28], check whether τ has associated vanishing even theta-null values. If so, terminate the algorithm.
- (iii) As in [28], construct a Weber model for τ , determine the corresponding invariants, and reconstruct a corresponding ternary quartic F .
- (iv) Calculate the period lattice Λ_F associated to F .
- (v) Using the methods from [8, §4.1], find a matrix T such that $T\Lambda_F = \Lambda R$ for some $R \in \mathrm{GL}_6(\mathbb{Z})$. Let $F_0 = F \cdot T^{-1}$ be the transform of F by the fractional linear transformation in the coordinates induced by T^{-1} .
- (vi) Calculate the period lattice Λ_0 for F_0 and (again using [8]) find μ be such that $\mu\Lambda = \Lambda_0$.
- (vii) Return the ternary quartic μF_0 .

The correctness of Algorithm 2.22 follows from the sources cited therein and Proposition 2.12.

Remark 2.23. The calculation of μ in Step (vi) is in fact superfluous, as the effect of applying T to G on the resulting period lattices can be described in terms of a power of its determinant. For ease of exposition, we have used the description above.

2.2. Interpolation. The results so far are purely numerical and specific to the base field \mathbb{C} . We now interpolate them to obtain explicit formulae. This process leads to very large formulae that we cannot display in this article, and in fact all of the considerations in this section will be descriptive rather than explicit. However, the process of obtaining the relevant formulae is documented in the ZIP file available at [25]. We sketch the main ideas.

We start with formal monic defining equations

$$X : y^2 = (x - \alpha_1) \cdots (x - \alpha_4) \tag{2.24}$$

and

$$Y : y^2 = (x - \beta_1) \cdots (x - \beta_4)(x^2 + ax + b). \tag{2.25}$$

We can then consider the gluing for the group G specified in the previous section, with β_5, β_6 the roots of the symmetrized polynomial $x^2 + ax + b$. This gives rise to a ternary quartic form $F(x, y, z)$ that defines a curve Z . While Proposition 2.12 shows that this equation is only determined up to a minus sign, the corresponding curve is still canonically determined as a subvariety of $\mathbb{P}_{\mathbb{C}}^2$. More concretely, we can obtain a unique normalized defining equation for Z by dividing by the coefficient of x^4 in F . We want to determine the dependence of this equation on the α_i and β_j , and will achieve this by a suitable interpolation process that we will prove correct a posteriori in Section 2.3.

It is far too ambitious to start working with all α_i and β_j simultaneously. Instead, we have gradually worked our way up. We sketch our procedure.

- (i) First we work with as few moduli parameters as possible and consider equations of the form

$$X : y^2 = x(x - 1)(x - \alpha) \tag{2.26}$$

and

$$Y : y^2 = x(x - 1)(x - \beta)(x^2 + ax + b) \tag{2.27}$$

That is, we take $\alpha_1 = \beta_1 = \infty$, $\alpha_2 = \beta_2 = 0$, $\alpha_3 = \beta_3 = 1$, $\alpha_4 = \alpha$, $\beta_4 = \beta$, while β_5, β_6 are the roots of $x^2 + ax + b$ as before.

We consider the monomials in α, β, a, b of degree at most 4. There are 70 of these. We therefore generate 200 quartics over \mathbb{Q} by taking random integer values for α, β, a, b between -10 and 10 , and apply the LLL algorithm to the result of Algorithm 2.22 in order to obtain good rational approximations numerical values for the coefficients of the normalized defining equation of Z . (Note that we know the resulting equation to be defined over \mathbb{Q} , since by our choice of α, β, a, b the curves X and Y as well as the gluing datum are defined over that field.)

We then try to find rational expressions in α, β, a, b for the coefficients of Z that interpolate these equations. It turns out that all instances are interpolated by the ternary quartic

$$\begin{aligned}
& (\alpha^2\beta^2 - \alpha^2\beta - \alpha\beta^2 + \alpha\beta)x^4 \\
& + (a\alpha^2\beta - a\alpha\beta^2 - a\alpha\beta + a\beta^2 + b\alpha^2 - 2b\alpha\beta + b\beta^2 + \alpha^2\beta - 2\alpha\beta^3 + 2\alpha\beta^2 - 2\alpha\beta + \beta^3)x^2y^2 \\
& + (-2a\alpha^2\beta^2 + 4a\alpha\beta^2 - 2a\beta^2 - 2b\alpha^2\beta + 2b\alpha\beta^2 + 2b\alpha\beta - 2b\beta^2 - 2\alpha^2\beta^2 + 2\alpha\beta^3 + 2\alpha\beta^2 - 2\beta^3)x^2yz \\
& + (a\alpha^2\beta^2 - a\alpha\beta^3 - a\alpha\beta^2 + a\beta^3 + b\alpha^2\beta - 2b\alpha\beta^3 + 2b\alpha\beta^2 - 2b\alpha\beta + b\beta^3 + \alpha^2\beta^2 - 2\alpha\beta^3 + \beta^4)x^2z^2 \\
& + (-a\alpha\beta^2 + a\alpha\beta + a\beta^3 - a\beta^2 - \alpha\beta^2 + \alpha\beta + \beta^4 - \beta^3)y^4 \\
& + (2a\alpha\beta^3 - 2a\alpha\beta^2 - 2a\beta^3 + 2a\beta^2 - 2b\alpha\beta^2 + 2b\alpha\beta + 2b\beta^3 - 2b\beta^2 + 2\alpha\beta^3 - 2\alpha\beta^2 - 2\beta^4 + 2\beta^3)y^3z \\
& + (a^2\alpha\beta^3 - a^2\alpha\beta^2 - a^2\beta^3 + a^2\beta^2 + ab\alpha\beta^2 - ab\alpha\beta - ab\beta^3 + ab\beta^2 + a\alpha\beta^3 - a\alpha\beta^2 \\
& \quad - a\beta^4 + a\beta^3 + 4b\alpha\beta^3 - 2b\alpha\beta^2 - 2b\alpha\beta - 2b\beta^4 - 2b\beta^3 + 4b\beta^2)y^2z^2 \\
& + (2ab\alpha\beta^3 - 2ab\alpha\beta^2 - 2ab\beta^3 + 2ab\beta^2 + 2b^2\alpha\beta^2 - 2b^2\alpha\beta - 2b^2\beta^3 + 2b^2\beta^2 - 2b\alpha\beta^3 + 2b\alpha\beta^2 + 2b\beta^4 - 2b\beta^3)yz^3 \\
& + (-ab\alpha\beta^3 + ab\alpha\beta^2 + ab\beta^4 - ab\beta^3 - b^2\alpha\beta^2 + b^2\alpha\beta + b^2\beta^4 - b^2\beta^3)z^4,
\end{aligned} \tag{2.28}$$

which is still of somewhat acceptable size — at least after simplifying by changing the factor in front of x^4 to $\alpha^2\beta^2 - \alpha^2\beta - \alpha\beta^2 + \alpha\beta$. Testing this result on a few thousand quartics more confirms it, completing the first step of our approach. We also observe that the new factor in front of z that clears denominators is nothing but a product of the discriminants of the polynomials $x(x-1)(x-\alpha)$ and $x(x-1)(x-\beta)$.

- (ii) The result in (i) already speeds up further considerations, since it obviates all but the final two steps in Algorithm 2.22, saving considerable calculation time. We now explore further by keeping one of (2.26) and (2.27) fixed and considering the general expression (2.24) and (2.25) for the other factor. At this point, we suspect that the resulting expressions are polynomial once the normalized defining equation for Z is multiplied with the product of the discriminants of $(x-\alpha_1)\cdots(x-\alpha_4)$ and $(x-\beta_1)\cdots(x-\beta_4)$. This turns out to be the case: the corresponding interpolation needs a few thousand curves but finds corresponding results with very small coefficients, though involving many monomials. We jot down the homogeneity degrees of these monomials in the α_i and β_j for later use.
- (iii) We now consider the equations (2.24) and (2.25) simultaneously, multiplying the normalized equation for Z with the same product of discriminants as before. Knowing what degrees of homogeneity in the α_i and β_j to expect for every defining coefficient cuts the number of candidate monomials down enormously, although it often remains considerable, in the order of several thousand at worst. However, finding a corresponding number of curves for interpolation is no problem, and with enough patience, the corresponding linear-algebraic calculations terminate. They again yield formulae with very modest coefficients (typically small powers of 2), but with a very large number of monomials.
- (iv) Having found these interpolated formulae, we can stress-test them further on several thousand more curves, until we are convinced that everything checks out.

Section 2.5 will mention a theoretical consideration that obviates the need for the later steps in the interpolation.

To summarize the heuristic results so far: Starting with X , Y and G at the beginning of this section, we have obtained a formula for a plane quartic

$$Z' : a_{400}x^4 + a_{220}x^2y^2 + a_{211}x^2yz + a_{202}x^2z^2 + a_{040}y^4 + a_{031}y^3z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4, \quad (2.29)$$

with the a_{ijk} polynomials in the α_i , β_j , a and b such that (conjecturally!) for generic values of the parameters, the resulting substitution yields a plane quartic whose Jacobian is isomorphic over \mathbb{C} to the quotient $(\text{Jac}(X) \times \text{Jac}(Y))/G$ with its induced principal polarization. Note the pleasant form of (2.29), which is a consequence of its canonicity.

Remark 2.30. We will not give a precise analysis of the degeneracy locus of the formulae obtained above. Since the discriminant of the resulting ternary quartic can be verified to be non-zero, it is generically well-defined. Its exact locus of definition is best studied in the context of a more detailed theoretical approach, perhaps by using the theory of algebraic theta functions, than the more ad hoc methods of this article.

Remark 2.31. While we have considered the formal equations (2.24) and (2.25) above, one also has to consider the cases where the defining polynomial of X and/or Y has odd degree. The resulting interpolation procedures are, however, completely similar, so we do not consider them further here or in what follows.

2.3. Rationality considerations and verification. Because we have determined our equation for a quotient by a subgroup of the 2-torsion in a canonical way, it stands to reason to expect that the resulting construction remains valid over base fields whose characteristic does not equal 2. This turns out to be the case. First, however, we will discuss how to prove when the heuristically interpolated equation (2.29) is actually correct, that is, when the Jacobian of the curve Z' in (2.29) actually splits as a product of the Jacobians of the given curves X and Y . We first consider these questions over \bar{k} and discuss the base field k in Section 2.4, where we will also generalize to the case where p_X and p_Y are not necessarily monic.

2.3.1. The genus-1 factor. The curve Z' has an obvious involution $\iota : (x, y, z) \mapsto (-x, y, z)$. We claim that the Jacobian of the corresponding quotient is generically indeed \bar{k} -isomorphic to $\text{Jac}(X)$.

Algorithm 2.32. This algorithm gives a method to verify the existence of a \bar{k} -isomorphism $\text{Jac}(Z'/\iota) \cong \text{Jac}(X)$.

Input: A curve X defined by an equation (2.1) and a curve Z' defined by an equation (2.29).

Output: A boolean that indicates whether there exists a \bar{k} -isomorphism $\text{Jac}(Z'/\iota) \cong \text{Jac}(X)$.

Steps:

- (i) Write (2.29) in the form $Ax^4 + Bx^2 + C$, with $A, B, C \in \bar{k}[y, z]$.
- (ii) Let $p = B^2 - 4AC$, and let p_0 be the homogenization of p_X .
- (iii) Let I, J (resp. I_0, J_0) be the binary quartic invariants of p (resp. p_0), as defined in [13].
- (iv) Check whether we have $(I : J) = (\mu^2 I_0, \mu^3 J_0)$ for some $\mu \in \bar{k}$. If so, return true, and return false otherwise.

The correctness of the algorithm follows from the observation that the quotient Z'/ι is isomorphic to the curve defined by p and the fact that given a binary quartic p with invariants I, J as in [13], the Jacobian of the curve corresponding to p is defined by $y^2 = x^3 - 27Ix - 27J$, see the first footnote in *loc. cit.*

Remark 2.33. The astute reader will note that Algorithm 2.32 does not work in characteristic 3. In this particular case, there are specific additional invariants (this time of weight 1 and 6 instead of 2 and 3) that can be used, as is explained and generalized to higher genus in Romain Basson's thesis [2]. We thank the referee for this reference.

Algorithm 2.32 is sufficiently simple to be run for the generic expression (2.29) (considered as a ternary quartic over a rational function field). It yields a positive response. The implementation at [24] still performs the corresponding check for every gluing that it constructs.

2.3.2. *The genus-2 factor.* It remains to check whether a complementary factor of the Jacobian of the curve Z' in (2.29) is given by the specified curve Y over \bar{k} . For this, we use the result [35] by Ritzenthaler and Romagny. We summarize their result in the following way:

Theorem 2.34 ([35]). *Consider a smooth plane quartic curve*

$$Z : x^4 + h(y, z)x^2 + f(y, z)g(y, z) = 0 \quad (2.35)$$

over k , where $h \in k[y, z]$ and $f, g \in \bar{k}[y, z]$ are binary quadratic forms. Let ι be the involution $\iota : (x, y, z) \mapsto (-x, y, z)$. Then there exists a polynomial $p(h, f, g) \in k[x]$ whose coefficients are polynomial expressions in those of h, f, g such that the Jacobian of the genus-2 hyperelliptic curve

$$Y : y^2 = p(h, f, g) \quad (2.36)$$

defines a degree-2 cover of the Prym variety of $Z \rightarrow Z/\iota$.

Remark 2.37. As mentioned in [35], the formulae from Theorem 2.34 only apply under certain genericity assumptions. We quietly pass over these in what follows, as we do with most results in this section.

We will not need the exact expression for p except for some formal transformation properties such as those in the upcoming Lemma. These can be verified by the reader using the procedures in [35]. The corresponding calculations are also checked at [24].

Lemma 2.38. *Denoting the substitution action of $A \in \mathrm{GL}_2(\bar{k})$ on a binary quadratic q by $q.A$, we have the following.*

- (i) $p(h, f, g) = p(h, g, f)$.
- (ii) $p(h, f, g) = p(h, \mu^{-1}f, \mu g)$ for all $\mu \in \bar{k}^*$.
- (iii) $p(h.A, f.A, g.A) = p(h, f, g).A'$, where $A' = UAU^{-1}$ for $U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proposition 2.39. *In the situation of Theorem 2.34, there exists a k -rational surjective map $\mathrm{Jac}(Z) \rightarrow \mathrm{Jac}(Y)$ if and only if the polynomial p has coefficients in k .*

Proof. The condition is certainly necessary, since k -rationality is not a meaningful notion otherwise. Suppose from now on that p has coefficients in k . One way to conclude is to invoke the canonicity of the constructions in [35], which ensure rationality over the base field whenever meaningfully possible. However, there is also a more direct proof.

First consider the special case where $f = f_0 := yz$. Then Z admits the k -rational point $P_0 = (0 : 0 : 1)$. To construct a k -rational map $\mathrm{Jac}(Z) \rightarrow \mathrm{Jac}(Y)$, it suffices to construct such a map for divisors of the form $[P] - [P_0] \in \mathrm{Jac}(Z)$. Moreover, by smoothness of the Jacobian, it in turn suffices to indicate the image of a generic point P . Given our equation for Y , we let ∞ be the degree-2 divisor of Y at infinity. We can then specify an element $D - \infty$ of $\mathrm{Jac}(Y)$ using the Mumford representation

$$\begin{aligned} 0 &= t^2 + a_1x + a_2 \\ y &= b_1x + b_2 \end{aligned} \quad (2.40)$$

of a degree 2 divisor D . Since P is a generic point and the map involved is non-constant, the coefficients a_1, a_2, b_1, b_2 are in the function field of Z .

The results at [24] contain an equation for a non-trivial divisor (2.40) that was obtained via another interpolation. It is defined over the base field. Moreover, it is verified that in terms of

the bases (2.8) and (2.9) the pullback of differentials is represented by the transpose of the matrix $T_0 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The latter property proves that the map $\text{Jac}(Z) \rightarrow \text{Jac}(Y)$ is surjective and that the corresponding factor of the Jacobian is complementary to the quotient by ι . Moreover, it is what we need to extend our results to general Z (on which there may not be a k -rational point, so that describing a map $\text{Jac}(Z) \rightarrow \text{Jac}(Y)$ becomes problematic).

Indeed, now let $F = x^4 + h(y, z)x^2 + f(y, z)g(y, z) \in k[x, y, z]$ with $f \in \bar{k}[y, z]$ be general. Then there exists $A \in \text{GL}_2(\bar{k})$ such that $f = f_0 \cdot A$. Let $F_0 = F \cdot \tilde{A}^{-1} \in \bar{k}[x, y, z]$, where $\tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Construct the polynomial p (resp. p_0) and the curve Y (resp. Y_0) corresponding to F (resp. F_0). The preceding special case describes the middle map in

$$\text{Jac}(Z) \rightarrow \text{Jac}(Z_0) \rightarrow \text{Jac}(Y_0) \rightarrow \text{Jac}(Y). \quad (2.41)$$

The map $\text{Jac}(Z) \rightarrow \text{Jac}(Z_0)$ is induced by the map of curves defined by \tilde{A} , and by Lemma 2.38(iii) there is an isomorphism $\text{Jac}(Y_0) \rightarrow \text{Jac}(Y)$ induced by $UA^{-1}U^{-1}$. The tangent representation of the composition (2.41) is therefore

$$\begin{aligned} UA^{-1}U \cdot T_0 \cdot \tilde{A} &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 0 & 2A \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = T_0. \end{aligned} \quad (2.42)$$

We conclude that the composition (2.41) still has tangent representation T_0 . Since the tangent representation uniquely determines the corresponding morphism, the latter is defined over k as well. \square

Given an equation Z of the form

$$Z : x^4 + h(y, z)x^2 + q(y, z) = 0 \quad (2.43)$$

there are 3 choices for the factorization in (2.35) over \bar{k} up to reordering and scaling. By Lemma 2.38(i) and (ii), the resulting equation for the curve Y in Theorem 2.34 depends only on this particular choice of partition of the roots of q into two pairs.

Proposition 2.44. *Given an equation (2.43) for a plane quartic curve Z , there exists a choice of factorization (2.35) that gives rise to a curve Y defined over k under the construction in Theorem 2.34 if and only if the cubic resolvent $\varrho(q)$ of q admits a root over k .*

Proof. This follows because the choice of a pair partition of roots of q uniquely determines a defining polynomial of Y . This polynomial is therefore defined over k , that is, Galois invariant, if and only if the given partition is. This is the case if and only if $\varrho(q)$ admits a root. Indeed, classical Galois theory shows that the splitting field of $\varrho(q)$ generates the subfield corresponding to the kernel of the conjugation action $\text{Gal}(q) \rightarrow \text{Aut}(V_4)$ on pairs of roots, and from the description of the roots in (1.44) we then see that the stabilizers of an individual root of $\varrho(q)$ is nothing but the stabilizer of a given pair partition. \square

Now consider the genus-2 curve Y with which we started. The matrix T_0 involved in Proposition 2.39 shows that if our results are correct, then we should expect the defining polynomial of p_Y of Y to coincide with the transformation $p_{Y'}(-x)$ of the polynomial defining one of the recovered factors Y' from Theorem 2.34. This turns out to be the case, but only up to a constant. In other words, we will have

$$p_Y(x) = cp_{Y'}(-x) \tag{2.45}$$

for some non-trivial $c \in k$. The reason for this phenomenon is the same as our reason for only considering isomorphisms over \bar{k} in Section 2.3.1, namely the presence of twists, a problem to which we turn in the next section. For now, we record our result over \bar{k} in the form of a criterion (whose conditions are of course far stronger than necessary, but are all that we need in our circumstances):

Algorithm 2.46. If the result of this algorithm is positive, then the Jacobian of Y is \bar{k} -isogenous to the complement of the image of $\text{Jac}(Z'/\iota)$ in $\text{Jac}(Z')$.

Input: A curve Y defined by an equation (2.2) and a curve Z' defined by an equation (2.29).

Output: A boolean that, if true, shows that there is a \bar{k} -isogeny between $\text{Jac}(Y)$ and the complement of the image of $\text{Jac}(Z'/\iota)$ in $\text{Jac}(Z')$.

Steps:

- (i) Rescale (2.29) to the form (2.35) and consider the three possible factorizations up to scalars $q = fg$ into binary quadratic factors.
- (ii) For each factorization pair, calculate a defining polynomial $p_{Y'}$ using Theorem 2.34. If for one of these polynomials we have $p_Y(x) = cp_{Y'}(-x)$ for some $c \in \bar{k}$, then return true. Otherwise return false.

In contrast to Algorithm 2.32, it costs a large amount of time and memory to run Algorithm 2.46 on the generic expression (2.29). Still, the implementation at [24] can rapidly perform the corresponding check for every concrete gluing that it constructs before returning its results.

2.4. Twists. Over an algebraically closed field k , positive results to the checks in Algorithms 2.32 and 2.46 previous sections suffice to demonstrate that for the curve Z' defined in (2.29), the Jacobian $\text{Jac}(Z')$ is indeed generically isogenous to $\text{Jac}(X) \times \text{Jac}(Y)$. We are now interested in doing likewise for general base field k , for which twists play a role, as mentioned in Theorem 1.57. The main problems in passing to the base field k are the following:

- (i) The isomorphism $\text{Jac}(Z'/\iota) \rightarrow \text{Jac}(X)$ in Algorithm 2.32 may be only defined over a proper extension of k ;
- (ii) The hyperelliptic curves defined by the polynomials p_Y and $p_{Y'}$ in Algorithm 2.46 may not be isomorphic over k .

In other words, when the necessary hypothesis of Proposition 2.39 is satisfied, we will have

$$\text{Jac}(Z') \sim \text{Jac}(X') \times \text{Jac}(Y') \tag{2.47}$$

for suitable curves X' and Y' of genus 1 and 2 over k , but X' and Y' need not be isomorphic to the specified curves X and Y over the base field k itself.

We are in Case (i) when $(I : J)$ and $(I_0 : J_0)$ are equivalent in weighted $(2, 3)$ -space over k , but not in weighted $(4, 6)$, whereas Case (ii) occurs when the scalar c in (2.45) is not a square in k . In either case, the issue is the presence of various quadratic twists.

We can resolve this problem and get rid of both twists by appropriately twisting both the curve Z' and its Jacobian. First, given a plane quartic curve Z , its Jacobian $\text{Jac}(Z)$ has the automorphism -1 , which does not come from an automorphism Z itself. We write $\mu * \text{Jac}(Z)$ to denote the quadratic twist of $\text{Jac}(Z)$ with respect to -1 by μ , as in Theorem 1.57.

Second, there is another twist available. If Z admits a defining equation (2.43), and $\nu \in k^*$, then we can take a quadratic twist of the curve Z itself with respect to the involution $\iota : (x, y, z) \mapsto (-x, y, z)$ by the scalar ν . This yields the curve

$$Z_\nu : \nu^2 x^4 + h(y, z)\nu x^2 + q(y, z) = 0. \quad (2.48)$$

Similarly, given a curve $X : y^2 = p_X$ of gonality 2, we write $X_\nu : y^2 = \nu p_X$ for its quadratic twist with respect to ι by the scalar ν .

Lemma 2.49. *Let Z be a genus-3 curve defined by an equation (2.43). Suppose that X and Y are such that $\text{Jac}(Z) \sim \text{Jac}(X) \times \text{Jac}(Y)$ over k . We have:*

- (i) $\mu * \text{Jac}(Z) \sim \text{Jac}(X_\mu) \times \text{Jac}(Y_\mu)$;
- (ii) $\text{Jac}(Z_\nu) \sim \text{Jac}(X) \times \text{Jac}(Y_\nu)$.

Proof. Part (i) follows because the projections $\text{Jac}(Z) \rightarrow \text{Jac}(X)$ and $\text{Jac}(Z) \rightarrow \text{Jac}(Y)$ commute with the automorphism -1 , which implies that over k the twist $\mu * \text{Jac}(Z)$ admits the factor $\mu * \text{Jac}(X) = \text{Jac}(X_\mu)$.

Part (ii) can be verified by explicit calculation. Indeed, the quotient

$$Z/\iota : x^2 + h(y, z)x + q(y, z) = 0 \quad (2.50)$$

is clearly k -isomorphic to

$$Z_\nu/\iota : \nu^2 x^2 + h(y, z)\nu x + q(y, z) = 0. \quad (2.51)$$

As for the new Y_μ , a direct calculation that its defining polynomial is ν^{-3} times that of Y , which implies the claim. \square

Since combining the actions in Lemma 2.49 allows us to twist both separate factors X' and Y' in any way desired, we can find μ and ν such that $\mu * \text{Jac}(Z'_\nu)$ has the requested factorization $\text{Jac}(X) \times \text{Jac}(Y)$ up to isogeny over the base field k .

This observation allows us equally well to deal with general defining equations (2.1) and (2.2) rather than merely their monic versions (2.24) and (2.25) that we used in our interpolation algorithms up until now. Moreover, the verification Algorithms 2.32 and 2.46 function equally well over k : In Algorithm 2.32, it suffices to check for equivalence of $(I : J)$ and $(I_0 : J_0)$ in weighted projective $(4, 6)$ -space over k instead of in weighted projective $(2, 3)$ -space over \bar{k} , and in Algorithm 2.46, it suffices to demand that c be a square in k .

Summarizing all that went before in this section, we have therefore obtained the following main algorithm, and with it, the Main Theorem:

Algorithm 2.52. This algorithm finds gluings of genus-1 and genus-2 curves along their torsion over the base field.

Input: Equations $X : y^2 = p_X$ and $Y : y^2 = p_Y$ that define curves of genus 1 and 2 over k .

Output: A (possibly empty) list L of pairs (Z, μ) , where Z is a smooth plane quartic and where $\mu \in k^*$ is a constant such that $\mu * \text{Jac}(Z) \sim \text{Jac}(X) \times \text{Jac}(Y)$.

Steps:

- (i) Initialize the empty list L . For all quadratic factors q_Y of p_Y , let $r_Y = p_Y/q_Y$ and perform all next steps but the final one.
- (ii) Check if the splitting fields of p_X and q_Y are isomorphic. If so, consider labelings of roots of p_X and q_Y such that the Galois actions on the corresponding roots (1.44) of the quadratic resolvents are compatible. For all such labelings, perform all next steps but the final one.
- (iii) Construct the interpolated curve Z' as in (2.29). Check that the coefficients of Z' belong to k . Construct $X' = Z'/\iota$ as in Algorithm 2.32 and check that X' is a quadratic twist of X with respect to -1 by μ say. Check that the cubic resolvent of $q = a_{040}y^4 + a_{031}y^3z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4$ admits at least one root over k , and that for one of the roots of

this resolvent, we have $p_Y(x) = cp(-x)$ for some $c \in k$, where p' is the defining polynomial for the curve Y' corresponding to the chosen root, as in Algorithm 2.46.

- (iv) Let $\nu = c\mu$, and let $Z = Z'_\nu$, so that $\mu * \text{Jac}(Z) = \mu * \text{Jac}(Z'_\nu) \sim \text{Jac}(X'_\mu) \times \text{Jac}(Y'_{\mu\nu}) \cong \text{Jac}(X'_\mu) \times \text{Jac}(Y'_c) \cong \text{Jac}(X) \times \text{Jac}(Y)$ by Lemma 2.49. Append (Z, μ) to L .
- (v) Return the list L .

Remark 2.53. The algorithms at [24] apply a more precise version of the above results, which considers the effect of twists on defining equations (2.1) (and hence on bases of differentials (2.7)) rather than on isomorphism classes. We omit the calculations, which only slightly refine the twisting scalars μ and ν involved, and describe the result. Recall from Section 2.1 that given defining equations for the curves X , Y , and Z , we can consider the bases of $\text{Jac}(Z)$ (resp. $\text{Jac}(X) \times \text{Jac}(Y)$) that correspond to the bases (2.9) and the union of the pullbacks of (2.7) and (2.8).

Now let defining polynomials p_X and p_Y for the input curves X and Y be given. Then the formulas at [24] give a ternary quartic equation F and a constant $\mu \in k$ such that there exists a map $\text{Jac}(Z) \rightarrow \text{Jac}(X) \times \text{Jac}(Y)$ with tangent representation $4\sqrt{\mu}$ with respect to the bases corresponding to p_X , p_Y , and F . Thus $\mu * \text{Jac}(Z)$ is isogenous over k to the product $\text{Jac}(X) \times \text{Jac}(Y)$. The factor 4 is inserted because it makes both F and μ smaller without losing integrality.

In practice, choosing this completely canonical approach gives rise to very agreeable expressions for Z and μ , especially when p_X and p_Y are in reduced minimized form. For example, the simple equations in (2.73), were found using this method, without any further optimization or reduction being needed.

2.5. Avoiding the larger interpolation. Recall from Section 2.2 that if X and Y are of the special form

$$X : y^2 = x(x-1)(x-\alpha) \tag{2.54}$$

and

$$Y : y^2 = x(x-1)(x-\beta)(x^2+ax+b) \tag{2.55}$$

then the interpolation gives rise to the simpler formula (2.28). Moreover, in this special case the twisting scalar μ with respect to -1 is given by

$$\begin{aligned} \mu = & \beta(\beta-1)(\beta-\alpha)(a^2\alpha^2\beta^2 - 2a^2\alpha\beta^2 + a^2\beta^2 + 2ab\alpha^2\beta - 2ab\alpha\beta^2 - 2ab\alpha\beta \\ & + 2ab\beta^2 + 2a\alpha^2\beta^2 - 2a\alpha\beta^3 - 2a\alpha\beta^2 + 2a\beta^3 + b^2\alpha^2 - 2b^2\alpha\beta + b^2\beta^2 \\ & + 2b\alpha^2\beta - 4b\alpha\beta^3 + 4b\alpha\beta^2 - 4b\alpha\beta + 2b\beta^3 + \alpha^2\beta^2 - 2\alpha\beta^3 + \beta^4). \end{aligned} \tag{2.56}$$

Since these expressions are so much smaller those involved in (2.29), the verification of their correctness via Algorithms 2.32 and 2.46 for the former is far less time-intensive than that for the latter. Moreover, this section will show that we can both reduce the latter generic check to the former and circumvent all but the first step in the interpolation in Section 2.2. For this, we use the following lemmata.

Lemma 2.57. *Let $k \subset \mathbb{C}$, and let X, Y, Z be defined by equations (2.1), (2.2), (2.4). Let $A \in \text{GL}_2(k)$, and let $X \cdot A$ (resp. $Y \cdot A$) be the curve with defining equation $p_X \cdot A$ (resp. $p_Y \cdot A$). Similarly, let $B \in \text{GL}_3(k)$, and let $Z \cdot B$ be the curve defined by $F \cdot B$. We have the following for the period matrices with respect to the canonical bases (2.7), (2.8), and (2.9):*

- (i) $\Lambda_X = \det(A)\Lambda_{X \cdot A}$;
- (ii) $\Lambda_Y = \det(A)A(\Lambda_{Y \cdot A})$;
- (iii) $\Lambda_Z = \det(B)B(\Lambda_{Z \cdot B})$.

Proof. The proof follows by considering the transformation of the bases of differentials (2.7), (2.8), and (2.9) under the specified substitutions. This follows via direct calculation, but also by first observing that the statements hold projectively by our canonical choices of differential bases and

then using diagonal matrices to determine the correct power of the determinant that is needed for actual equality. (We need only consider such powers since GL_n has no other characters.) \square

A similar argument shows the following.

Lemma 2.58. *Using the notation from the previous Lemma and Lemma 2.49, we have*

$$\begin{aligned} (i) \quad & \Lambda_X = \mu^{1/2} \Lambda_{X_\mu}; \\ (ii) \quad & \Lambda_Y = \mu^{1/2} \Lambda_{Y_\mu}; \\ (iii) \quad & \Lambda_Z = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu^{1/2} & 0 \\ 0 & 0 & \mu^{1/2} \end{pmatrix} \Lambda_{Z_\mu}. \end{aligned}$$

Furthermore, when identifying global differential forms of Z with translation-invariant differential forms on $\mathrm{Jac}(Z)$, we have that $\Lambda_Z = \mu^{1/2} \Lambda_{\mu^* \mathrm{Jac}(Z)}$.

We return to consider the gluing of the generic equations (2.24) and (2.25) from Section 2.2. There exist rational expressions α, β , matrices A, B , and scalars λ, μ in terms of the original formal parameters such that for the corresponding curves $X' = X(\alpha)$ and $Y' = Y(\beta)$ as in (2.54) and (2.55) we have

$$p_X = \lambda(p_{X'} \cdot A), \quad p_Y = \mu(p_{Y'} \cdot B). \quad (2.59)$$

Let Z' and the twisting scalar ν' with respect to ι describe the gluing of X' and Y' considered above, and define Z and ν for X and Y similarly. Since the gluing data coincide, we have

$$\begin{aligned} \Lambda_X \times \Lambda_Y &= \Lambda_{\nu^* \mathrm{Jac}(Z)} R, \\ \Lambda_{X'} \times \Lambda_{Y'} &= \Lambda_{\nu'^* \mathrm{Jac}(Z')} R \end{aligned} \quad (2.60)$$

for the same integral matrix R . The lemmata above for the hyperelliptic case show that

$$\Lambda_{X'} \times \Lambda_{Y'} = \begin{pmatrix} \lambda^{1/2} \det(A) & 0 \\ 0 & \mu^{1/2} \det(B) \end{pmatrix} \Lambda_X \times \Lambda_Y. \quad (2.61)$$

Therefore the same relation holds between $\Lambda_{\nu'^* \mathrm{Jac}(Z')}$ and $\Lambda_{\nu^* \mathrm{Jac}(Z)}$. Using the lemmata above in the plane quartic case shows that the gluing of X and Y is described by the curve Z and the twisting scalar ν with respect to ι , where

$$Z = (Z' \cdot T)_\chi, \quad \text{with } T = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \in \mathrm{GL}_3(k) \quad (2.62)$$

$$\text{and } \chi, \nu \text{ such that } \chi\nu = \mu \text{ and } \chi^2\nu = \lambda \det(A)^2 \det(B)^{-2}.$$

Since, as in Section 2.2, this construction ensures compatibility with the canonical bases of differentials, the equations from this new approach are exactly the same as those obtained before. However, this more subtle inroad has the advantage of proving the correctness of the complicated final expression along the way from that of the simpler case considered at the beginning of this section.

Note that, as was mentioned before, for every concrete input, our algorithms at [24] still perform a check of correctness regardless before returning the output, so that incorrect results are never returned.

2.6. A crucial symmetrization. A symmetrization of the formulae obtained above that is important in practice is the following. It is unpleasant, when starting with polynomials p_X and p_Y defining X and Y , to have to determine their roots α_i and β_j , as is currently required to apply the first formula (2.29). This leads to the determination of a compositum of the splitting field of two quartic polynomials, which already over \mathbb{Q} involves difficult field arithmetic that cannot be

circumvented by reduction algorithms like PARI's `polredabs` since the extensions involved may have degree larger than 20.

Instead, we use the more symmetric presentations

$$p_X = (x - \alpha_1) \cdots (x - \alpha_4) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \quad (2.63)$$

and

$$p_Y = (x - \beta_1) \cdots (x - \beta_4)(x^2 + ax + b) = (x^4 + b_1x^3 + b_2x^2 + b_3x + b_4)(x^2 + ax + b). \quad (2.64)$$

The considerations from Section 1 show that our chosen gluing only depends on a pairing of the roots $\gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, $\gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$ of the cubic resolvent of p_X with the roots $\delta_1, \delta_2, \delta_3$ of the cubic resolvent of $x^4 + b_1x^3 + b_2x^2 + b_3x + b_4$. We therefore expect that the coefficients a_{ijk} in (2.29) are invariant under the Klein Vierergruppen of permutations of α_i and β_j that stabilize all of the γ_i and δ_j . This indeed turns out to be the case.

Invariant theory shows that the polynomial expressions in $\alpha_1, \dots, \alpha_n$ that are invariant under the distinguished Vierergruppe are polynomial expressions in the invariants $\gamma_1, \gamma_2, \gamma_3$ of weight 2 and the coefficients a_1 and a_3 of p_X , which are of weight 1 and 3, respectively. A similar result of course holds for β_1, \dots, β_4 . We obtain the following result.

Proposition 2.65. *Starting with equations (2.24) and (2.25), the coefficients of the interpolated polynomial Z in (2.29) can be expressed as polynomials in the compositum of the splitting fields of the cubic resolvents of p_X and p_Y . In particular, if these splitting fields coincide (a necessary and sufficient condition for a gluing over the base field to exist by Theorem 1.51), then explicitly determining the coefficients of Z only requires intermediate calculations in this common splitting field of a cubic polynomial.*

Remark 2.66. Formally writing down the invariant expressions Proposition 2.65 already cuts down their length by a factor almost 20.

In practice, the formulae obtained in this way determine gluings over finite base fields (not necessarily prime fields) in a fraction of a second, whereas gluing curves over the rationals whose defining coefficients have about 100 decimal digits needs a bit over a minute (and results in defining coefficients with about 1600 decimal digits). Corresponding test suites are available at [24].

Remark 2.67. When considering the case where the curve Y admits a defining equation of the form $Y : y^2 = p(x^2)$, our algorithms directly recover the Ciani form (0.2). When $\text{Jac}(Y)$ is itself a 2-gluing of elliptic curves E_1 and E_2 , embedding X into its Jacobian allows use to recover maps $X \rightarrow E_1$ and $X \rightarrow E_2$ of degree 4 by functoriality.

2.7. Examples.

Example 2.68. Consider the genus-1 curve defined by the equation

$$X : y^2 = 4x^3 + 5x^2 - 98x + 157 = p_X. \quad (2.69)$$

It is isomorphic to the elliptic curve with label [118.c1](#) in the LMFDB [37]. Similarly, let

$$Y : y^2 = x^6 + 2x^3 - 4x^2 + 1 = p_Y. \quad (2.70)$$

be the genus-2 curve isomorphic to the curve in the LMFDB with label [295.a.295.1](#).

First we consider these curves in light of Section 1. The defining polynomial p_Y of Y factors as

$$(x - 1)(x^2 + x - 1)(x^3 + 2x + 1). \quad (2.71)$$

We see that there is a unique quadratic factor $q_Y = x^2 + x - 1$, with complement $r_Y = x^4 - x^3 + 2x^2 - x - 1$, and hence a unique subgroup $H \subset V_Y$ of dimension 1 that is Galois-stable. The cubic resolvent of r_Y is given by

$$\varrho(r_Y) = x^3 - 2x^2 + 5x - 8. \quad (2.72)$$

This defines the same splitting field as p_X : in fact both polynomials already define a common number field, which is isomorphic to that defined by $x^3 + 2x - 1$. Note that we have not taken a resolvent of p_X , in line with Remark 1.52.

The common splitting field of p_X and $\varrho(r_Y)$ has Galois group S_3 . Remark 1.50 shows that there is a single Galois equivariant isomorphism $\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$ for H . Since H itself was also unique, we conclude that $V_X \times V_Y$ has a single Galois stable maximal isotropic subgroup G .

The algorithms of Section 2 show that the quotient (Q, λ_Q) is a twist with respect to -1 by 5 of the Jacobian $(\text{Jac}(Z), \lambda_Z)$ of the plane quartic curve

$$Z : 32x^4 + 3x^2y^2 - 132x^2yz + 37x^2z^2 + 3y^4 - 14y^3z + 7y^2z^2 - 6yz^3 - 2z^4 = 0. \quad (2.73)$$

(More precisely, the twisting scalar μ with respect to -1 from Remark 2.53 is given by 5^3 .)

The LMFDB tells us that the Jacobian $\text{Jac}(X)$ has a rational 5-torsion point, and that $\text{Jac}(Y)$ has a 14-torsion point. As the isogeny defined by G has degree that is a power of 2, we can conclude that (Q, λ_Q) has a rational 70-torsion point if we show that the Galois module

$$W = (\text{Jac}(X)[2](\bar{k}) \times \text{Jac}(Y)[2](\bar{k}))/G \quad (2.74)$$

has a Galois stable subspace of dimension 1. For this, we use our knowledge of the subgroup G . The splitting field L of $p_X p_Y$ is of degree 12 over the base field $k = \mathbb{Q}$. We can label the roots $\alpha_1 = \infty, \alpha_2, \dots, \alpha_4$ of p_X and β_1, \dots, β_6 of p_Y in such a way that (i) $\beta_1 = 1$, (ii) β_5 and β_6 correspond to the quadratic factor q_Y , and (iii) the Galois action on the 2-torsion points $[\alpha_2] - [\alpha_1], [\alpha_3] - [\alpha_1], [\alpha_4] - [\alpha_1]$ in $\mathcal{G}(\mathcal{P})$ coincides with that on the elements $[\beta_2] - [\beta_1], [\beta_3] - [\beta_1], [\beta_4] - [\beta_1]$ of $\mathcal{G}(\mathcal{R})$ defined by r_Y .

Now consider the bases $\{[\alpha_2] - [\alpha_1], [\alpha_3] - [\alpha_1]\}$ of V_X and $\{[\beta_2] - [\beta_1], [\beta_3] - [\beta_1], [\beta_5] - [\beta_6], [\beta_5 - \beta_1]\}$ of V_Y . A calculation (performed in [24]) shows that the right action of two generators of $\text{Gal}(L|k)$ is given by the matrices

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.75)$$

Because of our ordering of the roots, the subgroup G corresponds to the subspace U given by

$$U = \langle (1 \ 0 \ 1 \ 0 \ 0 \ 0), (0 \ 1 \ 0 \ 1 \ 0 \ 0), (0 \ 0 \ 0 \ 0 \ 1 \ 0) \rangle. \quad (2.76)$$

This subspace is indeed stable under the action of the matrices above. If we take the images of the standard basis vectors e_2, e_3, e_6 as a basis for the corresponding quotient $W = V/U$, then the induced Galois action on W is described by the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (2.77)$$

There is a single non-trivial vector that is fixed under this action, which is the image of $e_6 = [\beta_5] - [\beta_1]$. Indeed the Galois action sends $[\beta_5] - [\beta_1]$ either to itself or to $[\beta_6] - [\beta_1]$, which is equivalent to $[\beta_5] - [\beta_1]$ modulo G , since the latter group contains the generator $[\beta_5] - [\beta_6]$ of H .

We have therefore shown that the twist of the Jacobian of the curve (2.73) with respect to -1 by 5 indeed contains a rational 70-torsion point.

Example 2.78. The complex-analytic reconstruction techniques in the previous section also allow one to construct gluings along 3-torsion. While it is more difficult to find examples of such gluings over the base field, one can still inspect which quotients by overlattices have invariants that are

numerically in the base field. For example, consider the case $k = \mathbb{Q}$ and the elliptic curve with LMFDB label [675.d2](#)

$$X : y^2 = 4x^3 + 25 \tag{2.79}$$

together with the genus-2 curve

$$Y : y^2 = x^5 + 20x^3 + 36x \tag{2.80}$$

which is a twist of the curve in the LMFDB with label [2916.b.11664.1](#). Over $\overline{\mathbb{Q}}$, the curves X and Y admit gluings along 3-torsion whose invariants are in \mathbb{Q} , as is shown in the example files at [\[24\]](#). Two such gluings are given by the base extensions of

$$6x^4 - 27x^2y^2 + 42x^2yz + 13x^2z^2 - 18y^4 - 30y^3z + 12y^2z^2 + 24yz^3 + 16z^4 = 0 \tag{2.81}$$

and

$$14x^3z + 3x^2y^2 - 210xyz^2 + 10y^3z + 1225z^4 = 0. \tag{2.82}$$

The former of these curves has its full endomorphism ring defined over a number field of degree 12, whereas the latter requires an extension of degree 18. Exact verification of these numerical results above is possible via [\[12\]](#). In fact, these also show that the curve Y is itself isogenous to a product of elliptic curves.

The algorithms at [\[24\]](#) also allow for the direct gluing of threefold products of elliptic curves along 3-torsion (or 2-torsion), as shown in the example files at [\[24\]](#).

Example 2.83. A final example is given by considering the curves

$$X : y^2 = x^4 + 2x^3 + x + 1 \tag{2.84}$$

and

$$Y : y^2 = 2x^6 + x^4 + x^3 + x^2 + 2x + 1 \tag{2.85}$$

over \mathbb{F}_3 . Our algorithms give two rise to two gluings, defined by the equations

$$Z_1 : x^4 + x^2yz + 2x^2z^2 + 2y^3z + y^2z^2 + z^4 = 0 \tag{2.86}$$

and

$$Z_2 : x^4 + 2x^2yz + x^2z^2 + 2y^3z + y^2z^2 + z^4 = 0. \tag{2.87}$$

For the former, the twisting scalar with respect to -1 is trivial, whereas the second requires a quadratic twist with respect to -1 by the non-square $-1 \in \mathbb{F}_3$ to recover the relevant abelian quotient variety.

3. GLUING VIA THE KUMMER VARIETY

In this section we describe a geometric algorithm that allows us to construct gluings of the curves X and Y over any algebraically closed base field k . The algorithm we describe (Algorithm [3.88](#)) reverses the construction of Ritzenthaler and Romagny in [\[35\]](#), mentioned above in Theorem [2.34](#).

3.1. Non-hyperelliptic curves of genus 3 that are gluings. We start by showing that every non-hyperelliptic curve of genus 3 can be given in the form of Theorem [2.34](#).

Lemma 3.1. *Let (Z, φ) be a gluing of X and Y over k . Then there is a degree-2 map $Z \rightarrow X$ over k .*

Proof. Dualizing the gluing map φ gives a map $\varphi^t : \text{Jac}(Z) \rightarrow \text{Jac}(X) \times \text{Jac}(Y)$. Choosing base points on Z and X , we obtain an inclusion $Z \rightarrow \text{Jac}(Z)$ and an identification $\text{Jac}(X) \cong X$. We can compose to obtain a non-constant map $f : Z \rightarrow X$. The corresponding pullback map $f^* : \text{Jac}(X) \rightarrow \text{Jac}(Z)$ is obtained by composing the canonical inclusion $\text{Jac}(X) \hookrightarrow \text{Jac}(X) \times \text{Jac}(Y)$ with φ . Because of the defining property of φ , the principal polarization λ_Z satisfies $(f^*)^*(\lambda_Z) \equiv 2\lambda_X$, which implies that f is of degree 2 by [\[5, Lemma 12.3.1\]](#). \square

Proposition 3.2. *Suppose that Z is a gluing of X and Y . If there exists a degree 2 morphism $\pi_2 : Z \rightarrow Y$, then Z is hyperelliptic.*

Proof. There exists a degree 2 morphism $\pi_1 : Z \rightarrow X$ by Proposition 3.1. Both π_1 and π_2 induce involutions of Z , which we denote i_1 and i_2 , respectively, and which in turn induce automorphisms i_1^* and i_2^* on $\text{Jac}(Z)$. Note that i_1 fixes divisor classes pulled back from X and i_2 fixes divisor classes pulled back from Y . Thus, by making a suitable choice of basis, we can represent i_1^* and i_2^* by the matrices

$$[i_1^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad [i_2^*] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (3.3)$$

in $\text{End}_k^0(\text{Jac}(Z)) \cong \text{End}_k^0(\text{Jac}(X)) \times \text{End}_k^0(\text{Jac}(Y))$. Letting $i = i_1 \circ i_2$, then

$$[i^*] = [i_1^* \circ i_2^*] = [i_1^*][i_2^*] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (3.4)$$

Thus i induces the negation map -1 on $\text{Jac}(Z)$, so Z is hyperelliptic by [29, Appendice, Théorème 3]. \square

Proposition 3.5. *With notation as in Theorem 2.34, the curve Z is a gluing of the curves Y and X .*

Proof. Consider the degree-2 cover $p : Z \rightarrow X$ given in Theorem 2.34, where $X = Z/\iota$. The map p induces an inclusion $p^* : (\text{Jac}(X), \lambda_X) \rightarrow (\text{Jac}(Z), \lambda_Z)$ of polarized abelian varieties, and by [5, Lemma 12.3.1] we find that

$$(p^*)^* \lambda_Z = 2\lambda_X, \quad (3.6)$$

the pullback of p^* .

Now $\text{Jac}(Y)$ is isomorphic to the Prym of an allowable singular cover $p' : Z' \rightarrow X'$ whose normalization is equal to $p : Z \rightarrow X$ as is shown in the proof of Theorem 1.1 in [35].

By [3, Theorem 3.7], the principal polarization on the generalized Jacobian $\text{Jac}(\tilde{Z})$ restricts to $2\lambda_Y$ on $\text{Pr}(Z'/X')$ where λ_Y is the principal polarization on $\text{Pr}(Z'/X') \cong \text{Jac}(Y)$. From [14, Lemma 1] we get a commutative diagram of polarized abelian varieties:

$$\begin{array}{ccc} \text{Pr}(Z'/X') & \xrightarrow{\nu} & \text{Pr}(Z/X) \\ \downarrow & & \downarrow i \\ \text{Jac}(Z') & \longrightarrow & \text{Jac}(Z) \end{array} \quad (3.7)$$

where ν is induced by the normalization $Z \rightarrow Z'$. This implies that

$$(i \circ \nu)^*(\lambda_Z) = 2\lambda_Y. \quad (3.8)$$

Now consider the map $j : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Z)$ defined by $(x, y) \mapsto p^*(x) + (i \circ \nu)(y)$. As $j = (i \circ \nu) \circ \pi_Y$ on the restriction to $\{0\} \times \text{Jac}(Y)$ we get that

$$j^*(\lambda_Z)|_{\{0\} \times \text{Jac}(Y)} = (\pi_Y^* \circ (i \circ \nu)^*(\lambda_Z))|_{\{0\} \times \text{Jac}(Y)} = \pi_Y^*(2\lambda_Y)|_{\{0\} \times \text{Jac}(Y)} \quad (3.9)$$

An analogous argument shows that

$$j^*(\lambda_Z)|_{\text{Jac}(X) \times \{0\}} = (\pi_X^* \circ (p^*)^*(\lambda_Z))|_{\text{Jac}(X) \times \{0\}} = \pi_X^*(2\lambda_X)|_{\text{Jac}(X) \times \{0\}}. \quad (3.10)$$

Using that our construction is generic, and that generically $\text{Hom}(A, B) = 0$ so that $\text{NS}(A) \times \text{NS}(B) \cong \text{NS}(A \times B)$, we can argue as in [7, Proposition 2.2] to conclude that $j^*(\lambda_Z)$ is algebraically equivalent to $2\pi_X^*(\lambda_X) \otimes 2\pi_Y^*(\lambda_Y)$, so Z is a gluing of $\text{Jac}(X)$ and $\text{Jac}(Y)$.

A more concrete approach in the case $k = \mathbb{C}$, which generalizes to arbitrary fields by the use of étale cohomology, is the following. The map p^* gives us an inclusion of $L_X = H_1(X, \mathbb{Z})$ into $L_Z = H_1(Z, \mathbb{Z})$ and the equality $(p^*)^* \lambda_Z = 2\lambda_X$ shows that the restriction of λ_Z to L_X gives us $2\lambda_X$. Now the kernel of the map p_* gives us the Prym variety $\text{Pr}(Z/X)$, which, using Lemma 1 of [14], comes equipped with a $(2, 1)$ -polarization that is the restriction of λ_Z to $\text{Pr}(Z/X)$. Let L_P be the sublattice of L_Z that corresponds to $\text{Pr}(Z/X)$. Then $p^*(L_X) \oplus L_P \subset L_Z$. The construction in [15, Theorem 4.2] of which [35] is the algebraic version, shows that the curve Y corresponds to a sublattice $q^*(L_Y)$ of index 2 in L_P on which the $(2, 1)$ -polarization restricts to a $(2, 2)$ -polarization. As a consequence, the restriction of λ_Z to $p^*(L_X) \oplus q^*(L_Y)$ is twice the product polarization, which ensures that the induced map $j : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Z)$ has the property that $j^*(\lambda_Z) = 2\pi_X^*(\lambda_X) \otimes 2\pi_Y^*(\lambda_Y)$ on $\text{Jac}(Z)$, which is what we wanted to show. \square

3.2. Plane sections of the Kummer. We begin by recalling some basic facts about Kummer surfaces. Throughout this section let X, Y , and Z be curves over k of genus 1, 2, and 3, respectively.

Definition 3.11. Let Y be a curve of genus 2 over k . The Kummer surface of Y is the quotient of $\text{Jac}(Y)$ by the negation map, i.e., $\text{Kum}(Y) = \text{Jac}(Y)/\langle -1 \rangle$.

Remark 3.12. Here and throughout we will abuse notation and denote by $\text{Kum}(Y)^t$ the Kummer surface of $\text{Jac}(Y)^t$, i.e., $\text{Kum}(Y)^t = \text{Jac}(Y)^t/\langle -1 \rangle$.

Proposition 3.13 ([18, Proposition 2.16]). *Let Y be a genus 2 curve. Then $\text{Kum}(Y)$ has 16 singular points, each one a node, and there exist 16 planes, called *special planes*, such that these planes and nodes form a nondegenerate $(16, 6)$ -configuration.*

Theorem 3.14. *Let Z be a non-hyperelliptic curve of genus 3 and let X be a curve of genus 1 such that we have a degree 2 cover $p : Z \rightarrow X$ which is ramified above exactly four points as in Theorem 2.34. Let Y be the curve of genus 2 given in this same theorem. Then there exist maps*

$$i_Z : Z \rightarrow \text{Jac}(Y)^t \text{ and } i_X : X \rightarrow \text{Kum}(Y)^t \quad (3.15)$$

of degree 1 such that the following diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{i_Z} & \text{Jac}(Y)^t \\ p \downarrow & & \downarrow \pi \\ X & \xrightarrow{i_X} & \text{Kum}(Y)^t \subset \mathbb{P}_k^3 \end{array} \quad (3.16)$$

Furthermore, there exists a plane $H \subset \mathbb{P}_k^3$ such that $i_X(X) = H \cap \text{Kum}(Y)^t$ and H contains exactly two singular points of $\text{Kum}(Y)^t$.

Proof. Using [1, 1.4] we see that the map p induces a commutative diagram

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \text{ker}(\text{Nm}) & & & \\ & & & \downarrow & \searrow e & & \\ 0 & \longrightarrow & \text{Jac}(X) & \xrightarrow{p^*} & \text{Jac}(Z) & \longrightarrow & \text{Pr}(Z/X)^t \longrightarrow 0 \\ & & \nearrow & & \downarrow \text{Nm} & & \\ & & Z & \xrightarrow{p} & X & & \\ & & & & \downarrow & & \\ & & & & 0 & & \\ & & & & 27 & & \end{array} \quad (3.17)$$

where

- (i) $\ker(\text{Nm})^0 = \text{Pr}(Z/X)$ is the Prym variety of p ,
- (ii) Nm is the norm map and $\text{Nm} \circ p^*$ is multiplication by 2.
- (iii) The map $j : Z \rightarrow \text{Pr}(Z/X)^t$ is an embedding.
- (iv) ϱ restricted to $\ker(\text{Nm})^0 \rightarrow \text{Pr}(Z/X)^t$ has 4 elements in its kernel and induces a (1,2)-polarization on $\text{Pr}(Z/X)$, which is given as a divisor by $j(Z)$.
- (v) The involution ι on Z defining the cover p extends to multiplication by -1 on $\text{Pr}(Z/X)^t$.

By [35, Lemma 3.3] we know that $\text{Pr}(Z'/X')$ is isomorphic to $\text{Jac}(Y)$ for a certain choice of a singular degree 2 cover $p' : Z' \rightarrow X'$ whose desingularization is equal to the original cover $p : Z \rightarrow X$. We furthermore remark that both X' and Z' have exactly two singular points, which are simple nodes and that these singular points coincide with the ramification points of the cover. Now using [14, Lemma 1] we get a map $\nu^* : \text{Pr}(Z'/X') \rightarrow \text{Pr}(Z/X)$ that sends the (1,2)-polarization to the principal polarization on $\text{Pr}(Z/X)$. Taking the dual of ν^* , we get a map $(\nu^*)^t : \text{Pr}(Z/X)^t \rightarrow \text{Pr}(Z'/X')^t = \text{Jac}(Y)^t$. Now define

$$i_Z : Z \rightarrow \text{Jac}(Y)^t$$

by $i_Z = (\nu^*)^t \circ j$. As j is an embedding by (iii) and $(\nu^*)^t$ is induced by the desingularization $Z \rightarrow Z'$ we get that $i_Z(Z)$ is isomorphic to Z' . As ι extends to multiplication by -1 on $\text{Pr}(Z/X)^t$ by (v), it naturally extends to multiplication by -1 on $\text{Jac}(Y)^t$. The singular points of $i_Z(Z)$ pass through the points that are invariant under this involution (as the singular points are the ramification points), so the singular points of Z' pass through the 2-torsion points of $\text{Jac}(Y)^t$. Now $j(Z)$ induces a (1, 2)-polarization on $\text{Pr}(Z/X)^t$ by (iv). Since ν^* sends the (1, 2)-polarization to a principal polarization, then $(\nu^*)^t$ sends $j(Z)$ to a divisor that is linearly equivalent to $2\Theta_Y$, where Θ_Y is the image of a theta divisor of Y under the principal polarization λ . By [18, Proposition 4.17] the map $\text{Jac}(Y)^t \rightarrow \text{Kum}(Y)^t$ is given by the linear system $|2\Theta_Y|$. This shows that $\pi(i_Z(Z))$ is given by $H \cap \text{Kum}(Y)^t$ for some plane H . As π commutes with p by construction there exists a map $i_X : X \rightarrow \text{Kum}(Y)^t$ such that the claim follows. \square

3.3. Geometric gluing data. Here we will give a geometric interpretation of the gluing data given in Corollary 1.37.

Lemma 3.18. *Let $\text{Kum}(Y)$ be a Kummer surface and let H be a special plane. Then $H \cap \text{Kum}(Y) = 2C$ where C is a conic. These conics, called **special conics**, are the only conics on $\text{Kum}(Y)$.*

Proof. See [18, Theorem 2.6] and [19, Proposition 2.18]. \square

Proposition 3.19. *Let $\text{Kum}(Y) = \text{Jac}(Y)/\langle -1 \rangle \subset \mathbb{P}_k^3$ be the Kummer surface associated to $\text{Jac}(Y)$ and let $\pi : \text{Jac}(Y) \rightarrow \text{Kum}(Y)$ be the quotient map. Fix a Weierstrass point Q on $\text{Jac}(Y)$ and let Θ_Y be the theta divisor given by the image of the map $Y \rightarrow \text{Jac}(Y), P \mapsto P - Q$.*

- (i) *Given $T \in \text{Jac}(Y)[2]$, then $\pi(t_T^*(\Theta_Y))$ is a special conic on $\text{Kum}(Y)$.*
- (ii) *Given a special conic C , then $C = \pi(t_T^*(\Theta_Y))$ for some $T \in \text{Jac}(Y)[2]$. Thus $\pi^*(C) = t_T^*(\Theta_Y)$, i.e., $\pi^*(C)$ is a translate of Θ_Y by a 2-torsion point on $\text{Jac}(Y)$.*

Proof. (i) Let Y_i be the curve given by $t_{T_i}^*(\Theta_Y)$ where $T_i \in \text{Jac}(Y)[2]$. The curve Θ_Y intersects $\text{Jac}(Y)[2]$ in exactly six points, so it follows that the curves Y_i also have this property as they are translates by a 2-torsion point. As π is a degree 2 map that is ramified only above the 2-torsion points, it follows by Riemann-Hurwitz that $\pi(Y_i)$ has genus 0. Since $\text{Kum}(Y)$ contains no lines (cf., [18, Theorem 4.30]), then $\pi(Y_i)$ must be a smooth conic on $\text{Kum}(Y)$. By Lemma 3.18 the special conics are the only conics on $\text{Kum}(Y)$, so $\pi(Y_i)$ has to be a special conic C . It follows that $\pi^*(C) = Y_i$, as desired.

(ii) Let S_1, \dots, S_6 be the six 2-torsion points that lie on Θ_Y . Then Y_i passes through the points $S_1 + T_i, \dots, S_6 + T_i$. A calculation shows that these sets of six 2-torsion points are distinct for all

sixteen $T_i \in \text{Jac}(Y)[2]$. It follows that the curves $\pi(Y_i)$ are sixteen distinct conics on $\text{Kum}(Y)$. As there are only sixteen special conics on $\text{Kum}(Y)$, the claim follows and we have proven (ii). \square

Lemma 3.20. *Let $\text{Kum}(Y)$ be as above and let H be a plane in \mathbb{P}_k^3 that intersects $\text{Kum}(Y)$ in exactly two singular points P_1, P_2 . Let $X' = H \cap \text{Kum}(Y)$. Then*

- (i) *The curve X' is a singular curve of genus 1 with exactly two nodes in P_1 and P_2 .*
- (ii) *There are exactly four distinct lines L_i in H such that $P_1 \in L_i, P_2 \notin L_i$ and X' is tangent to L_i . Furthermore, each of these L_i is the intersection of H with a special plane H_i that contains P_1 , but does not contain P_2 .*

Proof. (i) follows from [19, Proposition 2.20].

(ii) Any line through P_1 that is tangent to $\text{Kum}(Y)$ has to be contained in the intersection of the enveloping cone at P_1 and H . According to [18, Theorem 2.6] the enveloping cone of $\text{Kum}(Y)$ at P_1 consists of the six special planes of the Kummer surface that contain P_1 . Using the incidence relations of the (16,6) configuration [18, Lemma 1.7] we see that there are exactly four special planes H_i containing P_1 , but not P_2 . As H itself cannot be a special plane (as it contains only two singular points) it follows that the intersections $L_i = H \cap H_i$ give us four distinct lines that have the required properties. \square

Remark 3.21. The incidence relations also tell us there are only two special planes that contain both P_1 and P_2 .

Lemma 3.22. *Let $\text{Kum}(Y) \subset \mathbb{P}_k^3$ be a Kummer surface with singular points P_i . Let H be a plane that contains exactly two singular points P_1 and P_2 . Let $U \cong \mathbb{A}_k^2$ be an affine open of H containing both P_1 and P_2 . Let $\tilde{C} = \text{Kum}(Y) \cap U$. Let (x_i, y_i) be the coordinates of P_i in U , and define the function $\tilde{g} : \tilde{C} \setminus \{P_i\} \rightarrow k$ by*

$$\tilde{g}(x, y) = \frac{y - y_i}{x - x_i}. \quad (3.23)$$

Then \tilde{g} extends to a function

$$g : C \rightarrow \mathbb{P}_k^1 \quad (3.24)$$

of degree 2 where C is the normalization of \tilde{C} . Furthermore, the ramification points of g coincide with the slopes of the four lines L_i from Lemma 3.20.

Proof. Given a point $Q_1 = (x, y) \in \tilde{C}$, then $\tilde{g}(x, y)$ is exactly the slope of the line ℓ through P_1 and Q_1 . As $\text{Kum}(Y)$ is a quartic surface and U is a plane, then $\text{Kum}(Y) \cap U$ is a quartic plane curve. Then ℓ and \tilde{C} have intersection number 4 by Bézout's theorem. Since P_i is a node it contributes 2 to the intersection number, so ℓ generically intersects \tilde{C} in a third point Q_2 . Thus \tilde{g} is generically 2-to-1, hence has degree 2. The function \tilde{g} ramifies exactly when the the line ℓ is tangent to $\text{Kum}(Y)^t$, which occurs exactly when ℓ is the intersection of H with a special plane containing P_1 . As described in the proof of Lemma 3.20, exactly four of these do not contain P_2 and hence will remain ramification points of the map $g : C \rightarrow \mathbb{P}_k^1$ on the normalization. As C is a smooth genus 1 curve by 3.20(i) then the map g can have at most 4 ramification points. It follows that the ramification points of g coincide with the slopes of the four lines L_i . \square

Remark 3.25. We will later use the points in the reduced subscheme of the intersection of X' with a special plane to describe the 2-torsion points on $\text{Jac}(X')$.

Remark 3.26. Below we sometimes apply results stated for $\text{Jac}(Y)$ (resp., $\text{Kum}(Y)$) to $\text{Jac}(Y)^t$ (resp., $\text{Kum}(Y)^t$). This is permissible since $\text{Jac}(Y)$ is principally polarized: we have an isomorphism $\text{Jac}(Y) \xrightarrow{\sim} \text{Jac}(Y)^t$ that also induces an isomorphism $\text{Kum}(Y) \xrightarrow{\sim} \text{Kum}(Y)^t = \text{Jac}(Y)^t / \langle -1 \rangle$.

Lemma 3.27. *Let X' be as above and let X be the normalization of X' . Assume that $\text{Jac}(Y)^t$ is not isogenous to $\text{Jac}(X) \times \text{Jac}(E)$ for some elliptic curve E . Then $Z' = \pi^{-1}(X')$ is an irreducible singular curve of genus 3 with exactly two nodes.*

Proof. Assume that Z' is reducible and write $Z' = C' \cup D'$ for two curves C' and D' . As π is of degree 2, both C' and D' must be birational to X' . Since C' is birational to X' , then X is the normalization of C' , and hence the inclusion $C' \rightarrow \text{Jac}(Y)^t$ induces a surjective map $\text{Jac}(Y) \rightarrow \text{Jac}(X)$ (where we identify $(\text{Jac}(Y)^t)^t$ with $\text{Jac}(Y)$). This contradicts the assumption that $\text{Jac}(Y)^t$ is not isogenous to $\text{Jac}(X) \times \text{Jac}(E)$ for any E . When Z' is irreducible, it follows by [19, Proposition 2.20] that it is a singular curve of genus 3 with exactly two nodes. \square

Let Z' be a singular genus 3 curve whose singular points consist of exactly two nodes. Let $\nu : Z \rightarrow Z'$ be its desingularization. Then the pullback map ν^* induces an exact sequence

$$0 \rightarrow (k^*)^2 \rightarrow \text{Jac}(Z') \xrightarrow{\nu^*} \text{Jac}(Z) \rightarrow 0 \quad (3.28)$$

(cf., [30, Lemma 5.18]).

Proposition 3.29. *Let $\pi : \text{Jac}(Y)^t \rightarrow \text{Kum}(Y)^t$, X' , H and P_1 be as in Lemma 3.20. Let $S_1 \in \text{Jac}(Y)^t$ be the point that has $\pi(S_1) = P_1$. Let $Z' \subset \pi^{-1}(X')$ be an irreducible curve of genus 3. Let C_1, \dots, C_4 be the four special conics that pass through P_1 and intersect X' . Let $Y_j = \pi^{-1}(C_j)$, fix $\Theta_Y = Y_1$, and identify $T_j \in (\text{Jac}(Y)^t)^t[2] = \text{Jac}(Y)[2]$ with the divisor $Y_j - Y_1$ via the Abel-Jacobi map $P \mapsto t_P^*(Y_1) - Y_1$. Let R_j be the point in $(C_j \cap X')_{\text{red}}$ that is not equal to P_1 . Then there exists an isogeny $\varphi : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Z')$ such that $\varphi(\langle (R_j - R_1, T_j - T_1) \rangle) \subset \ker(\nu^*)$.*

Proof. Let $p' : Z' \rightarrow X'$ be the restriction of π to Z' . We now consider the pullback map $p'^* : \text{Jac}(X') \rightarrow \text{Jac}(Z')$. The morphism π is smooth outside of the 2-torsion points on $\text{Jac}(Y)^t$ and we can calculate p'^* by taking pre-images of points on the smooth locus. We can calculate the image of the 2-torsion of $\text{Jac}(X')$ by studying the pre-image of the divisors $R_j - R_1$. Let ι_j be the involution on Y_j . By construction $\pi^*(R_j - R_1) = Q_j + \iota_j(Q_j) - Q_1 - \iota_1(Q_1)$ for some points Q_j on Y_j .

On the other hand let $i_{Z'} : Z' \rightarrow \text{Jac}(Y)^t$ be the inclusion map. This induces a pullback map of divisors $i_{Z'}^* : \text{Jac}(Y) \rightarrow \text{Jac}(Z')$. The divisor $Y_j - Y_1$ corresponds via the polarization $P \mapsto t_P^*(Y_1) - Y_1$ to a 2-torsion point T_j on $\text{Jac}(Y)$.

As $C_j = \pi(Y_j)$ is tangent to X' in R_j and intersects the node P_1 , it follows that $Z' \cap Y_j = Q_j + \iota_j(Q_j) + 2S_1$ if we purely look at the underlying sets. Outside of the singular locus we may treat Cartier divisors as if they are Weil divisors. When we pull back the divisor $T_j \in \text{Jac}(Y)$ to $\text{Jac}(Z')$ by $i_{Z'}$, however, we may need to take the contribution of the singular point S_1 into account. It now follows that

$$i_{Z'}^*(T_j - T_1) \sim Q_j + \iota_j(Q_j) + s_1 - Q_1 - \iota_1(Q_1) - s_j \quad (3.30)$$

where the s_i have the property that $\nu^*(s_i) = W_{1,1} + W_{1,2}$ where the $W_{1,i}$ are the two points above S_1 in the normalization $\nu : Z \rightarrow Z'$. Consider the map $p'^* + i_{Z'}^* : \text{Jac}(X') \times \text{Jac}(Y) \rightarrow \text{Jac}(Z')$. By the above we find that

$$(p'^* + i_{Z'}^*)(R_j - R_1, T_j - T_1) = 2(Q_j + \iota_j(Q_j) - Q_1 - \iota_1(Q_1)) + s_j - s_1 \sim s_j - s_1 \quad (3.31)$$

as $Q_j + \iota_j(Q_j) - Q_1 - \iota_1(Q_1)$, being the image of a 2-torsion point, is a 2-torsion point. Furthermore,

$$\nu^*(s_j - s_1) = W_{1,1} + W_{1,2} - W_{1,1} - W_{1,2} \sim 0,$$

so we conclude that

$$(p'^* + i_{Z'}^*)(\langle (R_j - R_1, T_j - T_1) \rangle) \subset \ker \nu^*. \quad (3.32)$$

\square

Lemma 3.33. *Let $i_{Z'} : Z' \rightarrow \text{Jac}(Y)^\dagger$ be as above and write P_1 and P_2 for the nodes of Z' . Let T_5 and T_6 be the two points on $\text{Jac}(Y)$ such that $T_5 = Y_5 - Y_1$, $T_6 = Y_6 - Y_1$ and Y_5 and Y_6 are the two curves that intersect both P_1 and P_2 . Then $i_{Z'}^*(T_5 - T_6) \subset \ker \nu^*$.*

Proof. As points, we find that $Y_5 \cap Z' = Y_6 \cap Z' = 2P_1 + 2P_2$ by construction. Similarly to the above proposition we have

$$i^*(T_5 - T_6) = s_1 - s_2 \quad (3.34)$$

for some $s_1, s_2 \in \text{Jac}(Z')$ with the property that $\nu^*(s_i) = W_{1,1} + W_{1,2} + W_{2,1} + W_{2,2}$. (Similarly to what we did above the $W_{2,i}$ are the two points above S_2 in the normalization $\nu : Z \rightarrow Z'$). It follows that $i_{Z'}^*(T_5 - T_6) \subset \ker \nu^*$. \square

We write $p : Z \rightarrow X$ for the normalization of the singular cover $p' : Z' \rightarrow X'$.

Proposition 3.35. *The kernel of the gluing isogeny $\varphi : \text{Jac}(X) \times \text{Jac}(Y) \rightarrow \text{Jac}(Z)$ where $\varphi = (p^* + (i_{Z'} \circ \nu)^*)$ is given by*

$$\langle (R_j - R_1, T_j - T_1), (0, T_5 - T_6) \rangle. \quad (3.36)$$

Proof. As we may assume that Θ_X does not lie on the singular part of X' , we may assume that

$$\ker \varphi = \ker(p^* + (i_{Z'} \circ \nu)^*) = \ker \nu^* \circ (p'^* + i_{Z'}^*) \quad (3.37)$$

Using Lemma 3.29 and Lemma 3.33 we find that

$$\langle (R_j - R_1, T_j - T_1), (0, T_5 - T_6) \rangle \subset \ker \varphi. \quad (3.38)$$

Since $\ker(\varphi)$ is a maximal isotropic subgroup of $\text{Jac}(X)[2] \times \text{Jac}(Y)^\dagger[2]$, then $\ker \varphi$ is isomorphic to \mathbb{F}_2^3 as a vector space. As $(R_2 - R_1)$ and $(R_3 - R_1)$ are linearly independent in $\text{Jac}(X)[2]$, we see that $(R_2 - R_1, T_2 - T_1), (R_3 - R_1, T_3 - T_1)$ and $(0, T_5 - T_6)$ are linearly independent in $\text{Jac}(X)[2] \times \text{Jac}(Y)^\dagger[2]$. This completes the proof. \square

By Theorem 3.14 we know that every non-hyperelliptic genus 3 gluing Z can be mapped into the $\text{Jac}(Y)^\dagger$ via an immersion and Proposition 3.35 tells us how we can read off the gluing kernel. We can use this to see in which way the gluings show up.

Lemma 3.39. *Let P_1 and P_2 be nodes of $\text{Kum}(Y)$ and let $H(\lambda)$ be the family of planes going through P_1 and P_2 . Then the j -invariant of the family $H(\lambda) \cap \text{Kum}(Y)$ is a rational function $j(H(\lambda)) \in k(\lambda)$ of degree at most 12.*

Proof. By [18, Proposition 2.20] we may assume that K is given by the homogeneous polynomial

$$\kappa(x, y, z, t) = x^4 + y^4 + z^4 + t^4 + 2Dxyzt + A(x^2t^2 + y^2z^2) + B(y^2t^2 + x^2z^2) + C(z^2t^2 + x^2y^2) \quad (3.40)$$

in \mathbb{P}_k^3 with singular points $P_1 = (d, -c, b, -a)$ and $P_2 = (d, c, -b - a)$. In this case the family of planes going through P_1 and P_2 is given by

$$H(\lambda) = ax + by + cz + dt + \lambda(ax - by - cz + dt). \quad (3.41)$$

Without loss of generality we assume that $b, d \neq 0$. Let U be the affine open subset of $H_{1,2}(\lambda)$ that we get by setting $z = 1$ to get a plane that contains both P_1 and P_2 . Let $\tilde{C}_{\lambda_0} = U \cap K$. It follows that we can describe \tilde{C}_{λ_0} as a curve in \mathbb{A}_k^2 given by the equation $F(x, y) = 0$ where

$$F(x, y) = \kappa \left(x, y, 1, \frac{(1 + \lambda)ax + (1 - \lambda)(by + c)}{d(-1 - \lambda)} \right) \quad (3.42)$$

and define an isomorphism $\varphi : \tilde{C}_{\lambda_0} \rightarrow K \cap U$ by

$$\varphi(x, y) = \left(x, y, 1, \frac{(1 + \lambda)ax + (1 - \lambda)(by + c)}{d(-1 - \lambda)} \right). \quad (3.43)$$

Using this isomorphism we get $\varphi^{-1}(d, -c, b, -a) = (d/b, -c/b)$. Let

$$g : (U \cap K) \setminus \{(d/b, -c/b)\} \rightarrow k \quad (3.44)$$

be the function defined by mapping a point P to the slope of the line passing through $(d/b, -c/b)$ and P as in Lemma 3.22. We will find the ramification points of g to calculate the j -invariant of the family.

A line in U with slope μ passing through $(d/b, -c/b)$ satisfies the equation

$$y = \mu x - c/b - \mu d/b. \quad (3.45)$$

Consider the polynomial

$$F(x, \mu x - c/b - \mu d/b) \quad (3.46)$$

in $k(\lambda, \mu)[x]$.

Let $D(\mu) \in k(\lambda)$ be the discriminant of $F/((x - d/b)^2)$ with respect to x . Solving $D(\mu) = 0$ gives us the values of μ for which the intersection number of L with \tilde{C}_{λ_0} is greater than 2. We divide by $(x - d/b)^2$ to exclude the case where L intersects P_1 .

A calculation shows that the zeroes of $D(\mu)$ are:

$$0, \quad (3.47)$$

$$x_1(\lambda) = ((ab\lambda + ab + cd\lambda + cd)/(b^2\lambda - b^2 - d^2\lambda - d^2)), \quad (3.48)$$

$$x_2(\lambda) = ((ab\lambda + ab - cd\lambda - cd)/(b^2\lambda - b^2 + d^2\lambda + d^2)), \quad (3.49)$$

$$x_3(\lambda) = ((-ac\lambda - ac - bd\lambda - bd)/(ad\lambda + ad - bc\lambda + bc)), \quad (3.50)$$

$$x_4(\lambda) = ((-ac\lambda - ac + bd\lambda + bd)/(ad\lambda + ad - bc\lambda + bc)). \quad (3.51)$$

The 0 coincides with the horizontal line that passes through P_2 . The other values give us the branch points of the map g . Note that the $x_i(\lambda)$ are rational functions of degree 1 in λ . To compute the j -invariant of the normalization of \tilde{C}_{λ_0} we compute the cross-ratio of the $x_i(\lambda)$

$$c(\lambda) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \quad (3.52)$$

which is a rational function of degree at most 2. It follows that the j -invariant

$$j(\lambda) = \frac{(c(\lambda)^2 - c(\lambda) + 1)^3}{c(\lambda)^2(c(\lambda) - 1)^2}. \quad (3.53)$$

is a rational function in $k(\lambda)$ of degree at most 12. \square

Proposition 3.54. *Let X and Y be as above. Let G be a maximal isotropic subgroup of $\text{Jac}(X)[2] \times \text{Jac}(Y)[2]$ and (\mathcal{T}, ℓ) be the corresponding pair given by Proposition 1.18. Then choosing \mathcal{T} coincides with choosing two curves Y_5 and Y_6 on $\text{Jac}(Y)^\dagger$. Write P_1 and P_2 for the two singular points in $\pi(Y_5 \cap Y_6)$. Assuming we have done this, choosing ℓ is the same as picking a value of λ_0 in the family of planes $H_{1,2}(\lambda)$ such that $j(H_{1,2})(\lambda_0) = j(X)$. Moreover, for every choice of ℓ there generically exist two choices of λ_0 that have this property.*

Proof. Fix a 2-torsion point S_1 on $\text{Jac}(Y)^\dagger$. By the (16,6)-configuration there exist distinct divisors T_i with $i = 1, \dots, 6$ such that all Y_j (where $T_j = Y_j - Y_1$) contain the point S_1 . Remark that as the T_i are distinct, we have that $\langle T_i - T_j : i, j \in \{1, \dots, 6\} \rangle = \text{Jac}(Y)[2]$. Now let G be a maximal isotropic subgroup with corresponding pair (\mathcal{T}, ℓ) and assume that $\mathcal{T} = \{T_5, T_6\}$. It follows from the incidence relations of the (16, 6)-configuration that $Y_5 \cap Y_6$ consists of exactly two points. One of them is the S_1 we defined before and we will call the other one S_2 .

Consider the sets

$$\mathcal{P} = \{R_i \in \text{Jac}(X)[2] : i \in \{1, \dots, 4\}\}, \quad \mathcal{Q} = \{T_i \in \text{Jac}(Y) : i \in \{1, \dots, 6\}\}. \quad (3.55)$$

We remark that, analogously to Corollary 1.37, giving an indecomposable maximal isotropic subgroup of $\text{Jac}(X)[2] \times \text{Jac}(Y)[2]$ is the same as giving a subset \mathcal{T} of cardinality 2 along with a symplectic morphism $\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$ where $\mathcal{R} = \mathcal{Q} \setminus \mathcal{T}$.

Now define $P_i = \pi(S_i)$ for $i = 1, \dots, 2$ and let $H_{1,2}(\lambda)$ be the 1-dimensional family of planes in \mathbb{P}^3 parametrized by λ that contains both P_1 and P_2 . Using Lemma 3.39 we see that there exist 12 values λ_m such that $H_{1,2}(\lambda_m) \cap \text{Kum}(Y)^t$ for $m \in \{1, \dots, 12\}$ has the same j -invariant as X . To simplify notation we write $X'_m = H_{1,2}(\lambda_m) \cap \text{Kum}(Y)^t$. Generically, all of the planes in $H_{1,2}(\lambda)$ will contain exactly two singular points, so we may assume that the X'_m are genus 1 curves with exactly two nodes. Without loss of generality we may assume that R_j is the point in $\pi(Y_j) \cap X'_1$ that is not equal to P_1 .

Now let Z_1 be the normalization of the genus 3 component in $\pi^{-1}(X'_1)$. Using Proposition 3.5 and Theorem 3.14 we see that Z_1 is the gluing of X and Y with gluing datum (\mathcal{T}, ℓ_1) where $\mathcal{T} = \{T_5, T_6\}$ and where ℓ_1 is defined by $R_j \mapsto T_j$. Moreover, as the cross-ratio (3.52) is a function of degree 2, it will generically assume all six values that correspond to X twice. Choosing a different value for the cross-ratio permutes the R_j . This implies that for a given map $\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$ there exists exactly two m such that $\ell(T_j) = \pi(Y_j) \cap X'_m$ and such that Z_m is the gluing of X and Y with gluing datum $(\{T_5, T_6\}, \ell)$. \square

3.4. Making the construction explicit. Using the work of Mumford [33] and Cantor [9] on Jacobians of hyperelliptic curves, as well as the work of Müller [32] on Kummer surfaces, we will give explicit descriptions of the objects and maps used in Theorem 3.14.

We first recall how a divisor of degree 2 on a genus 2 curve can be represented in Mumford coordinates as a pair of polynomials.

Proposition 3.56. *Let Y be the affine part of a smooth curve of genus 2 over k given by a Weierstrass equation $y^2 = f(x)$ in \mathbb{P}_k^2 . Then there exists a bijection between the sets*

$$\mathcal{S} := \{(x_1, y_1), (x_2, y_2)\} \in \text{Sym}^2(Y) \mid x_1 \neq x_2\}$$

and

$$\mathcal{P} := \{(a(x), b(x)) \in k[x] \times k[x] \mid a \text{ is monic and separable, } \deg(a) = 2, \deg(b) \leq 1\}.$$

Proof. See [33, Proposition 1.2] or [9, §2]. \square

Given a pair $(a(x), b(x)) \in \mathcal{P}$, then $a(x) = x^2 + a_1x + a_2, b(x) = b_1x + b_2$ for some $a_1, a_2, b_1, b_2 \in k$. We can use these coefficients as coordinates on an open affine subset of $\text{Jac}(Y)$ as described in the following proposition.

Proposition 3.57. *Let Y be given by the equation $y^2 = f(x)$ in \mathbb{P}_k^2 . Let g_1 and g_2 be polynomials in $k[a_1, a_2, b_1, b_2]$ such that*

$$g_1(a_1, a_2, b_1, b_2)x + g_0(a_1, a_2, b_1, b_2) \equiv b(x)^2 - f(x) \pmod{a(x)}. \quad (3.58)$$

Then the system of equations

$$g_1(a_1, a_2, b_1, b_2) = 0, \quad (3.59)$$

$$g_2(a_1, a_2, b_1, b_2) = 0 \quad (3.60)$$

describes an affine open subset U of $\text{Jac}(Y)$ in \mathbb{A}_k^4 .

Proof. See Proposition 1.3 and Chapter IIIa, §2 of [33]. \square

Proposition 3.61. *Let Y be a curve of genus 2 over a field k given by the equation*

$$y^2 = f_0 + f_1x + f_2x^2 + f_3x^3 + f_4x^4 + f_5x^5 + f_6x^6 \quad (3.62)$$

in \mathbb{A}_k^2 . Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points on Y and let $P + Q \in U \subset \text{Jac}(Y)$ where U is as in Proposition 3.57. Let

$$\begin{aligned}\kappa_1 &= 1, \\ \kappa_2 &= x_1 + x_2, \\ \kappa_3 &= x_1 x_2, \\ \kappa_4 &= \frac{F_0(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2},\end{aligned}\tag{3.63}$$

where

$$F_0(x, y) = 2f_0 + f_1(x + y) + 2f_2(xy) + f_3(x + y)xy + 2f_4(xy)^2 + f_5(x + y)(xy)^2 + 2f_6(xy)^3.\tag{3.64}$$

Then we can define a map $\pi : U \rightarrow \text{Kum}(Y)$ given by $(P, Q) \mapsto (\kappa_1 : \kappa_2 : \kappa_3 : \kappa_4)$ such that π is equal to the quotient morphism $\text{Jac}(Y) \rightarrow \text{Kum}(Y)$ restricted to U .

The functions $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ satisfy the quartic equation

$$K(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = K_2(\kappa_1, \kappa_2, \kappa_3)\kappa_4^2 + K_1(\kappa_1, \kappa_2, \kappa_3)\kappa_4 + K_0(\kappa_1, \kappa_2, \kappa_3) = 0,\tag{3.65}$$

where

$$K_2(\kappa_1, \kappa_2, \kappa_3) = \kappa_2^2 - 4\kappa_1\kappa_3\tag{3.66}$$

$$\begin{aligned}K_1(\kappa_1, \kappa_2, \kappa_3) &= -4\kappa_1^3 f_0 - 2\kappa_1^2 \kappa_2 f_1 - 4\kappa_1^2 \kappa_3 f_2 - 2\kappa_1 \kappa_2 \kappa_3 f_3 \\ &\quad - 4\kappa_1 \kappa_3^2 f_4 - 2\kappa_2 \kappa_3^2 f_5 - 4\kappa_3^3 f_6\end{aligned}\tag{3.67}$$

$$\begin{aligned}K_0(\kappa_1, \kappa_2, \kappa_3) &= -4\kappa_1^4 f_0 f_2 + \kappa_1^4 f_1^2 - 4\kappa_1^3 \kappa_2 f_0 f_3 - 2\kappa_1^3 \kappa_3 f_1 f_3 \\ &\quad - 4\kappa_1^2 \kappa_2^2 f_0 f_4 + 4\kappa_1^2 \kappa_2 \kappa_3 f_0 f_5 - 4\kappa_1^2 \kappa_2 \kappa_3 f_1 f_4 - 4\kappa_1^2 \kappa_3^2 f_0 f_6 \\ &\quad + 2\kappa_1^2 \kappa_3^2 f_1 f_5 - 4\kappa_1^2 \kappa_3^2 f_2 f_4 + \kappa_1^2 \kappa_3^2 f_3^2 - 4\kappa_1 \kappa_2^3 f_0 f_5 \\ &\quad + 8\kappa_1 \kappa_2^2 \kappa_3 f_0 f_6 - 4\kappa_1 \kappa_2^2 \kappa_3 f_1 f_5 + 4\kappa_1 \kappa_2 \kappa_3^2 f_1 f_6 \\ &\quad - 4\kappa_1 \kappa_2 \kappa_3^2 f_2 f_5 - 2\kappa_1 \kappa_3^3 f_3 f_5 - 4\kappa_2^4 f_0 f_6 - 4\kappa_2^3 \kappa_3 f_1 f_6 \\ &\quad - 4\kappa_2^2 \kappa_3^2 f_2 f_6 - 4\kappa_2 \kappa_3^3 f_3 f_6 - 4\kappa_3^4 f_4 f_6 + \kappa_3^4 f_2^2,\end{aligned}\tag{3.68}$$

and equation (3.65) gives us a projective embedding of $\text{Kum}(Y)$ in \mathbb{P}_k^3 .

Proof. See [32, §2]. □

Corollary 3.69. Let U be an affine open subset of $\text{Jac}(Y)$ in $\mathbb{A}_k^4 = k[a_1, a_2, b_1, b_2]$ given by the system of equations $g_1 = 0, g_2 = 0$ as in Proposition 3.57. Then the map $U \rightarrow \text{Kum}(Y)$ described in Proposition 3.61 can be explicitly described as

$$(a_1, a_2, b_1, b_2) \mapsto \left(1 : -a_1 : a_2 : \frac{\tilde{F}_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2} \right)\tag{3.70}$$

where

$$\tilde{F}_0(x, y) = 2f_0 + f_1 x + 2f_2 y + f_3 xy + 2f_4 y^2 + f_5 xy^2 + 2f_6 y^3.\tag{3.71}$$

Proof. The correspondence described in Proposition 3.56 sends a pair of points $\{P_1, P_2\} \in \mathcal{S}$ with $P_i = (x_i, y_i)$, $i = 1, 2$, to the pair of polynomials $(a(x), b(x))$ with

$$a(x) = (x - x_1)(x - x_2) \quad \text{and} \quad b(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1.$$

The result now follows from equating coefficients and then substituting these expressions into the formula for F_0 given in Proposition 3.61. □

Remark 3.72. The point $(0 : 0 : 0 : 1)$ is always a singular point on the projective embedding of $\text{Kum}(Y)$ in \mathbb{P}_k^3 given by equation (3.65), as can be verified by a computation of partial derivatives.

Lemma 3.73. *Let $\varphi : k(\text{Kum}(Y)) \rightarrow k(\text{Jac}(Y))$ be the inclusion of function fields induced by the morphism $\pi : \text{Jac}(Y) \rightarrow \text{Kum}(Y)$. Then*

(i) *There exist $\alpha_i, \beta_j \in k(a_1, a_2)$ such that*

$$b_1 b_2 = \alpha_1(a_1, a_2) + \alpha_2(a_1, a_2) b_1^2, \quad (3.74)$$

$$b_2^2 = \beta_1(a_1, a_2) + \beta_2(a_1, a_2) b_1^2. \quad (3.75)$$

(ii) *Let*

$$h := \frac{(\kappa_2^2 - 4\kappa_3)\kappa_4 - \tilde{F}_0(\kappa_2, \kappa_3) + 2\kappa_2\alpha_1(-\kappa_2, \kappa_3) + 2\beta_1(-\kappa_2, \kappa_3)}{-2\kappa_3 - 2\kappa_2\alpha_2(-\kappa_2, \kappa_3) - 2\beta_2(-\kappa_2, \kappa_3)} \quad (3.76)$$

Then $\varphi(h) = b_1^2$.

Proof. Note that the polynomials g_1 and g_2 of Proposition 3.57 can be computed by dividing $b(x)^2 - f(x) = (b_1x + b_2)^2 - f(x)$ by $a(x)$ using polynomial long division, hence are contained in the subring $k[a_1, a_2][b_1^2, b_1b_2, b_2^2]$. Considering the system of equations $g_1 = g_2 = 0$ as linear equations in b_1^2, b_1b_2, b_2^2 over the field $k(a_1, a_2)$, we can solve for b_1b_2 and b_2^2 in terms of b_1^2 .

Recall from Corollary 3.69 that on the affine open of $\text{Jac}(Y)$ with coordinates a_1, a_2, b_1, b_2 we have $\varphi(\kappa_2) = -a_1$, $\varphi(\kappa_3) = a_2$, and

$$\varphi(\kappa_4) = \frac{\tilde{F}_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2}.$$

Solving for b_1b_2 and b_2^2 in terms of a_1, a_2 , and b_1^2 as in part (i), we can express $\varphi(\kappa_4)$ as a function of $\varphi(\kappa_2), \varphi(\kappa_3)$, and b_1^2 . A somewhat laborious computation then shows that

$$b_1^2 = \frac{(\varphi(\kappa_2)^2 - 4\varphi(\kappa_3))\varphi(\kappa_4) - \tilde{F}_0(\varphi(\kappa_2), \varphi(\kappa_3)) + 2\varphi(\kappa_2)\alpha_1(-\varphi(\kappa_2), \varphi(\kappa_3)) + 2\beta_1(-\varphi(\kappa_2), \varphi(\kappa_3))}{-2\varphi(\kappa_3) - 2\varphi(\kappa_2)\alpha_2(-\varphi(\kappa_2), \varphi(\kappa_3)) - 2\beta_2(-\varphi(\kappa_2), \varphi(\kappa_3))}.$$

Thus for h as defined as in the statement of the lemma, we have $\varphi(h) = b_1^2$. \square

Corollary 3.77. *Let π, φ and h be as in Lemma 3.73. Then we can extend φ to a morphism $\bar{\varphi} : k(\text{Kum}(Y))[\sqrt{h}] \rightarrow k(\text{Jac}(Y))$ such that $\bar{\varphi}$ is an isomorphism. Furthermore, let C be a curve on $\text{Kum}(Y)$, let $k(C)$ be the function field of C , and suppose that $D = \pi^{-1}(C) \subset \text{Jac}(Y)$ is irreducible. Then $k(C)[\sqrt{h}]$ is the function field of D .*

Proof. Define $\bar{\varphi}(f + g\sqrt{h}) = \varphi(f) + \varphi(g)b_1$. As a_1, a_2, b_1, b_2 are the coordinates of a dense affine open subset of $\text{Jac}(Y)$, it suffices to show that they are in the image of $\bar{\varphi}$. From Corollary 3.69 we have $a_1 = -\bar{\varphi}(\kappa_2)$, $a_2 = \bar{\varphi}(\kappa_3)$ and $\bar{\varphi}(\sqrt{h}) = b_1$, and from equation (3.74) we have

$$b_2 = \frac{\alpha_1(a_1, a_2)}{b_1} + \alpha_2(a_1, a_2)b_1$$

so b_2 is also in the image of $\bar{\varphi}$.

The final statement follows from the fact that the inclusion of function fields $k(C) \hookrightarrow k(C)[\sqrt{h}]$ corresponds to the morphism of curves $D \rightarrow C$. \square

Remark 3.78. The condition that $D = \pi^{-1}(C)$ be irreducible is satisfied in the context, as Theorem 3.14 shows.

The following result is a slight generalization of [15, Theorem 1.1].

Proposition 3.79. *Let C be a genus 1 curve over an algebraically closed field k with $\text{char}(k) \neq 2$, and let P_1, P_2, P_3, P_4 be distinct points in C . Then*

- (i) There are exactly four distinct covers $D_j \rightarrow C$ (where $j = 1, \dots, 4$) of degree 2 that are ramified above the P_i and unramified everywhere else.
- (ii) If there exists a function $f \in k(C)$ such that $k(D_i) \cong k(C)[\sqrt{f}]$ for some i , then for each $j = 1, \dots, 4$ there exists a degree-0 divisor T (representing a non-trivial element of $\text{Pic}(C)[2]$) such that D_j is isomorphic to a curve with function field $k(C)[\sqrt{f_T}]$ with $\text{div}(f_T) = \text{div}(f) + 2T$.
- (iii) Let T_1, \dots, T_4 be the order-2 Weierstrass points on C . There exist non-constant functions $u \in L(T_1 + T_i)$ and $v \in L(2T_1 + 2T_i)$ with $\text{div}(v) = P_1 + P_2 + P_3 + P_4 - 2T_1 - 2T_i$ such that
 - (a) The curve C has an equation of the form

$$v^2 + vh(u) + f(u) = 0 \quad (3.80)$$

where h is a polynomial of degree 2 and f is a polynomial of degree 4.

- (b) The curve D_i has an equation over k of the form

$$t^4 + t^2h(s) + f(s) = 0 \quad (3.81)$$

where h is a polynomial of degree 2 and f is a polynomial of degree 4.

- (c) The cover $\pi_i : D_i \rightarrow C$ is explicitly given by $\pi_i(s, t) = (s, t^2)$.

Proof. See [22, Paragraph 4.4]. □

Finally, we give an explicit expression for the map i_Z from Theorem 3.14.

Proposition 3.82. *For a non-hyperelliptic curve Z of genus 3 that is a gluing of X and Y the code in [36] contains an explicit description of the map $i_Z : Z \dashrightarrow \text{Jac}(Y)^t$ from Theorem 3.14.*

Proof. According to Proposition 3.1 the gluing Z gives rise to a cover as in Theorem 2.34. The code in [36] contains an explicit rational map $Z \dashrightarrow \text{Jac}(Y)^t$. After a change of coordinates we may assume that Z has an affine open V of the form

$$v^4 + v^2g(u) + uh(u) = 0 \quad (3.83)$$

where $g(u) = g_2u^2 + g_1u + g_0$ and $h(u) = h_2u^2 + h_1u + h_0$. We calculate an equation for Y using 2.34 and use this equation to construct the affine open $U \subset k[a_1, a_2, b_1, b_2]$ of the Jacobian $\text{Jac}(Y)^t$ given by the equations in Proposition 3.57.

Let

$$\begin{aligned} \alpha(u, v) &= (g_2h_0 - g_0h_2)v^2 + (g_2^2h_0 - g_2g_0h_2)u^2 + (g_2g_1h_0 - g_2g_0h_1 - h_2h_0)u, \\ \beta(u, v) &= g_2^2h_0 - g_2g_0h_2v^3 + (g_2^3h_0 - g_2^2g_0h_2)u^2 \\ &\quad + ((g_2^2g_1h_0 - g_2g_1g_0h_2 - g_2h_2h_0 + g_0h_2^2)u + g_2^2g_0h_0 - g_2g_0^2h_2)v \\ N(u, v) &= (g_2^2h_1 - g_2g_1h_2 + h_2^2)u + g_2^2h_0 - g_2g_0h_2. \end{aligned} \quad (3.84)$$

Then the map $i_Z : V \rightarrow U$ is explicitly given by

$$(u, v) \mapsto (\alpha(u, v)/N(u, v), 0, \beta(u, v)/(N(u, v)), \beta(u, v)/(uN(u, v))). \quad (3.85)$$

In the code it is shown that the image of i_Z is generically contained in U . A proof for the generic injectivity of the map $i_Z : Z \rightarrow \text{Jac}(Y)^t$ due to Davide Lombardo is the following: Assume that i_Z is not injective. If $i_Z(Z)$ is of genus 2, then Z would be hyperelliptic by Proposition 3.2, which gives us a contradiction, so $i_Z(Z)$ is either of genus 1 or of genus 0. It is impossible for $i_Z(Z)$ to be of genus 0, as then by the theory of abelian varieties the map i_Z would be constant. On the other hand, if $i_Z(Z)$ is a curve of genus 1 then $\text{Jac}(Y)^t$ would be isogenous to the product of two elliptic curves, which cannot be true generically.

Let $\pi : U \rightarrow \text{Kum}(Y)^t$ be the map given in Corollary 3.69. As $i_Z(Z)$ is contained in the plane given by $a_2 = 0$, it follows that $\pi(i_Z(Z))$ is a curve contained in the plane H defined by $\kappa_3 = 0$. This yields a rational map $Z \dashrightarrow \pi(i_Z(Z))$ of degree 2. We claim that the curve $\pi(i_Z(Z))$ is of genus 1. Indeed, if $\pi(i_Z(Z))$ is not of genus 1 then it will either be of genus 2 or of genus 0. But in both

of these cases Z would be a hyperelliptic curve. Indeed, if $\pi(i_Z(Z))$ has genus 2 then Proposition 3.2 tells us that Z is a hyperelliptic curve. In the second case we have a degree 2 cover from Z to a genus 0 curve, so the statement follows by definition. We therefore conclude that $\pi(i_Z(Z))$ is of genus 1. As any plane section of a quartic surface in \mathbb{P}_k^3 has arithmetic genus 3 this means that the plane H has to intersect $\text{Kum}(Y)^t$ in two singular points. Finally, it remains to be shown that the above diagram commutes. Let $i : Z \rightarrow Z$ be the involution $(u, v) \mapsto (u, -v)$ that corresponds to the degree 2 cover $Z \rightarrow X$. Then the explicit form of the maps at [36] shows that

$$\begin{aligned} i_Z(i(u, v)) &= i_Z((u, -v)) \\ &= (\alpha(u, -v)/N(u, -v), 0, \beta(u, -v)/(N(u, -v)), \beta(u, -v)/(-vN(u, -v))) \\ &= (\alpha(u, v)/N(u, v), 0, -\beta(u, v)/(N(u, v)), -\beta(u, v)/(uN(u, v))). \end{aligned} \quad (3.86)$$

Now the map $(a_1, a_2, b_1, b_2) \mapsto (a_1, a_2, -b_1, -b_2)$ sends the divisor $P + Q$ corresponding to the equations $x^2 + a_1x + a_2$ and $y = b_1x + b_2$ to the divisor $P' + Q'$ corresponding to the equations $x^2 + a_1x + a_2$ and $y = -b_1x - b_2$. Therefore $i_Z \circ i$ is multiplication by -1 on $\text{Jac}(Y)^t$ and we conclude that we have found a commutative diagram as in (3.16). \square

3.5. The algorithm. In this section we combine the previous results into an algorithm for computing all non-hyperelliptic gluings of a genus 1 and a genus 2 curve.

Remark 3.87. The following algorithms may require extensions of the base field in order to produce explicit equations.

Algorithm 3.88. This algorithm constructs all gluings of X and Y that produce non-hyperelliptic curves.

Input: Curves $X : y^2 = p_X$ and $Y : y^2 = p_Y$ of genera 1 and 2, respectively, and a pair $(T_5 - T_6, \ell)$ as in Proposition 1.18.

Output: A degree 2 cover $p : Z \rightarrow X$ such that Z is a gluing of X and Y with gluing datum G , where G is the maximal isotropic subgroup corresponding to $(T_5 - T_6, \ell)$.

Steps:

- (1) Calculate an affine model for $\text{Jac}(Y)^t$ as in Corollary 3.57.
- (2) Compute $j(X)$.
- (3) Calculate a model for $\text{Kum}(Y)^t$ and the projection map $\text{Jac}(Y)^t \rightarrow \text{Kum}(Y)^t$ as in Proposition 3.61.
- (4) Calculate a function h with the property that $k(\text{Kum}(Y)^t)[\sqrt{h}] \cong k(\text{Jac}(Y)^t)$ as in Lemma 3.73.
- (5) Let P_1 be the singular point $(0 : 0 : 0 : 1)$ on $\text{Kum}(Y)^t$ and let Θ_Y be a theta divisor such that $\pi(\Theta_Y)$ passes through P_1 .
- (6) Determine P_2 such that $\pi(Y_5) = \pi(\Theta_Y)$ and $\pi(Y_6)$ intersect both $(0 : 0 : 0 : 1)$ and P_2 .
- (7) Calculate the 1-dimensional family $H_{1,2}(\lambda)$ of planes that pass through P_1 and P_2 .
- (8) Calculate the set $\Lambda(X)$ of all λ such that $j(H_{1,2}(\lambda) \cap \text{Kum}(Y)^t) = j(X)$.
- (9) For each $\lambda_0 \in \Lambda(X)$:
 - (a) Calculate the points R_j where $R_j \in \pi(Y_j) \cap X'$ and $R_j \neq P_1$.
 - (b) If $\ell(T_j - T_1)$ is linearly equivalent to the divisor $R_j - R_1$ on X' for all j .
 - (i) Calculate the curve Z with function field $k(\tilde{X}(\lambda_0))[\sqrt{h}]$ using Algorithm 3.90; this gives us the desired gluing and a natural projection map to X .
 - (ii) Return the calculated gluing Z .

Remark 3.89. By the (16,6)-configuration we can uniquely identify the divisors Y_j in the algorithm by choosing six 2-torsion points on $\text{Jac}(Y)$.

Algorithm 3.90. This algorithm calculates an equation for the curve Z with function field $k(\tilde{X}(\lambda_0))[\sqrt{h}]$.

Input:

- A singular genus-1 curve $\tilde{X}(\lambda_0)$ with exactly two nodes that is given by the intersection of a plane and a quartic surface.
- A function $h \in k(\tilde{X}(\lambda_0))$ whose divisor is of the form $P_1 + P_2 + P_3 + P_4 - 2T$ for some divisor T .

Output: A degree 2 cover $p : Z \rightarrow X(\lambda_0)$ such that $k(Z) \cong k(\tilde{X}(\lambda_0))[\sqrt{h}]$ where $X(\lambda_0)$ is the normalization of $\tilde{X}(\lambda_0)$.

Steps:

- (1) Use Lemma 3.22 to compute the branch points $\alpha_1, \dots, \alpha_4$ of the map $g : \tilde{X}(\lambda_0) \rightarrow \mathbb{P}_k^1$ that maps a point P to the slope of the line that connects P with the singular point $(1 : 0 : 0 : 0)$.
- (2) Define the curve C by the equation $y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ and compute a birational map $\tau : C \rightarrow \tilde{X}(\lambda_0)$.
- (3) Let Q, R be the singular points on $\tilde{X}(\lambda_0)$ and calculate $\tau^{-1}(Q) = \{Q_1, Q_2\}$ and $\tau^{-1}(R) = \{R_1, R_2\}$.
- (4) Calculate the divisor D of the image of h in $k(C)$.
- (5) Find divisors D_1, D_2, D_3, D_4 that correspond to the four distinct degree 2 covers with ramification points Q_1, Q_2, R_1, R_2 as in Proposition 3.79.
- (6) For each $i \in \{1, \dots, 4\}$:
 - (a) If there exists a principal divisor T such that $D_i - D = 2T$, then:
 - (i) Calculate the degree 2 cover $Z \rightarrow X$ corresponding to D_i using Riemann-Roch spaces which is possible according to Proposition 3.79(iii).
 - (ii) Return a quartic equation of Z along with the map of degree 2 to X .

Example 3.91. We illustrate the field extensions required by the second method. Let X be the curve given by

$$y^2 = x^4 + 2x^3 - x^2 - 2x \quad (3.92)$$

and let Y be the curve given by

$$y^2 = x^6 - 2x^5 - 10x^4 + 20x^3 + 9x^2 - 18x \quad (3.93)$$

over \mathbb{Q} . An affine open of $\text{Jac}(Y)^t$ is given by the following system of equations in $\mathbb{Q}[a_1, a_2, a_3, a_4]$.

$$\begin{aligned} -a_1^4 a_2 - 2a_1^3 a_2 + 3a_1^2 a_2^2 + 10a_1^2 a_2 + 4a_1 a_2^2 + 20a_1 a_2 - a_2^3 - 10a_2^2 + a_2 b_1^2 - 9a_2 - b_2^2 &= 0, \\ -a_1^5 - 2a_1^4 + 4a_1^3 a_2 + 10a_1^3 + 6a_1^2 a_2 + 20a_1^2 - 3a_1 a_2^2 - 20a_1 a_2 + a_1 b_1^2 - 9a_1 - 2a_2^2 - 20a_2 - 2b_1 b_2 - 18 &= 0. \end{aligned} \quad (3.94)$$

The equation for $\text{Kum}(Y)^t$ in $\mathbb{P}_{\mathbb{Q}}^3$ is

$$\begin{aligned} 324x_1^4 + 720x_1^3 x_3 - 720x_1^2 x_2 x_3 - 144x_1 x_2^2 x_3 + 72x_2^3 x_3 + 832x_1^2 x_3^2 - 36x_2^2 x_3^2 + 80x_1 x_3^3 - 80x_2 x_3^3 + 44x_3^4 + \\ 36x_1^2 x_2 x_4 - 36x_1^2 x_3 x_4 - 40x_1 x_2 x_3 x_4 + 40x_1 x_3^2 x_4 + 4x_2 x_3^2 x_4 - 4x_3^3 x_4 + x_2^2 x_4^2 - 4x_1 x_3 x_4^2 &= 0 \end{aligned} \quad (3.95)$$

and the morphism $\pi : \text{Jac}(Y)^t \rightarrow \text{Kum}(Y)^t$ is explicitly given by

$$\pi(a_1, a_2, b_1, b_2) = \left[1 : -a_1 : a_2 : \frac{2a_1 a_2^2 - 20a_1 a_2 + 2a_1 b_1 b_2 + 18a_1 + 2a_2^3 - 20a_2^2 - 2a_2 b_1^2 + 18a_2 - 2b_2^2}{a_1^2 - 4a_2} \right]. \quad (3.96)$$

We consider the family of planes $H_{1,2}(\lambda)$ passing through the singular points $P = (0 : 0 : 0 : 1)$ and $Q = (-1/6 : 1/3 : 1/2 : 1)$. The j -invariant of X is $35152/9$, and we seek to find the values

of λ such that $H_{1,2}(\lambda) \cap \text{Kum}(Y)^t$ has the same j -invariant. We calculate the j -invariant $j(\lambda)$ of $H_{1,2}(\lambda) \cap \text{Kum}(Y)$ and find that the numerator of $j(\lambda) - 35152/9$ factors as

$$\left(\lambda - \frac{9}{23}\right) \left(\lambda - \frac{1}{11}\right) \left(\lambda^2 - \frac{38}{67}\lambda - \frac{9}{67}\right) \left(\lambda^2 - \frac{98}{193}\lambda - \frac{3}{193}\right) \left(\lambda^2 - \frac{42}{85}\lambda + \frac{1}{85}\right) \left(\lambda^2 - \frac{22}{47}\lambda + \frac{3}{47}\right) \left(\lambda^2 - \frac{2}{5}\lambda + \frac{1}{5}\right). \quad (3.97)$$

We will construct the degree 2 cover above $\tilde{X}(9/23) = H_{P,Q}(9/23) \cap \text{Kum}(Y)^t$. (The other covers can be computed in the same way by choosing other roots of (3.97). To compute $k(\tilde{X}_1(-3/2))/(t^2 - h)$ we proceed as in Algorithm 3.90 and calculate the branch points of the degree 2 map $g : \tilde{X}(9/23) \rightarrow \mathbb{Q}$. We compute the Legendre model $y^2 = x(x-1)(x-1/4)$ for $\tilde{X}(9/23)$, which we denote \tilde{X}_{leg} . Although \tilde{X}_{leg} is a nontrivial twist of $\tilde{X}(9/23)$ over \mathbb{Q} , the curves become isomorphic upon extending scalars to $\mathbb{Q}(\alpha)$, where α is a root of

$$t^2 - 156026658225043557710221401/34308279913908709968852208000.$$

We explicitly compute the image of the function h in the function field of $k(\tilde{X}_{\text{leg}}) \otimes \mathbb{Q}(\alpha)$ and obtain a rational function of degree 14 with rather large coefficients, which we denote h_{leg} .

Let P_1, P_2 (resp., Q_1, Q_2) be the two points obtained by desingularizing P (resp., Q). We find that the divisor $P_1 + P_2 + Q_1 + Q_2$ is defined over the field $\mathbb{Q}(\beta, \gamma)$ where β is a root of $t^2 + 3/32$ and γ is a root of $t^2 - 327/250t + 4761/10000$. Let T_1, \dots, T_4 be the 2-torsion points of \tilde{X}_{leg} . For each $i = 1, \dots, 4$, we calculate functions f_i with $\text{div } f_i = P_1 + P_2 + Q_1 + Q_2 - 2T_i - 2T_1$ for $i = 1, \dots, 4$. We determine which f_i corresponds to our covering $Z \rightarrow X$ by checking if $\text{div}(f_i/h_{\text{leg}})$ is divisible by 2 for each i . Applying Riemann-Roch as in Proposition 3.79 we compute the equation

$$\begin{aligned} u^4 - \frac{244312307247680}{12491063134299} \alpha u^3 + \left(\frac{286830015625}{36438849216} \beta \gamma - \frac{250115773625}{48585132288} \beta \right) u^2 v^2 + \frac{5876}{8855} u^2 \\ + \left(\frac{-50500786167745625000}{1338579798660883737} \alpha \beta \gamma + \frac{11009171384568546250}{446193266220294579} \alpha \beta \right) uv^2 - \frac{83804221642880}{37473189402897} \alpha u \\ - \frac{1044509681265625}{171408346712064} v^4 + \left(\frac{52518171875}{63767986128} \beta \gamma - \frac{45795845875}{85023981504} \beta \right) v^2 + \frac{1460}{111573} = 0. \end{aligned} \quad (3.98)$$

for the gluing over $\mathbb{Q}(\alpha, \beta, \gamma)$. A simplified equation of this curve over \mathbb{Q} , found by the methods in Section 2, is

$$12x^4 - 111x^2y^2 + 478x^2yz - 577x^2z^2 - 533y^4 + 948y^3z - 2574y^2z^2 + 2196yz^3 - 2277z^4 = 0. \quad (3.99)$$

Over finite fields, the current Kummer method is less involved, since coefficient explosion is excluded and the required field extensions are less complicated.

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