BELYI MAPS AND DESSINS D'ENFANTS
HOMEWORK #3

Exercise 3.1. Let $C : X^3 + Y^3 = Z^3$ and define

$$f : C \to \mathbb{P}^1$$
$$[X : Y : Z] \mapsto [X : Z] .$$

Determine the ramification points of f and their multiplicities.

Exercise 3.2.

(a) Let $f \in \mathcal{M}(X)$ be a nonconstant meromorphic function on a compact Riemann surface X. Show that $\sum_P \mathrm{ord}_P(f) = 0$. (*Hint*: Use results about degree.)

(b) Given $0 \neq x \in \mathbb{Z}$ and a prime $p \in \mathbb{Z}$, define the p-adic valuation

$$\mathrm{ord}_p(x) := \max\{v \in \mathbb{Z} : p^v \mid x\}$$

and extend the definition to $x = a/b \in \mathbb{Q}$ by

$$\mathrm{ord}_p(x) = \mathrm{ord}_p(a/b) := \mathrm{ord}_p(a) - \mathrm{ord}_p(b) .$$

By convention, we define $\mathrm{ord}_p(0) = \infty$. Define the p-adic absolute value by

$$|x|_p = p^{-\mathrm{ord}_p(x)}$$

for $x \in \mathbb{Q}$. Finally, define $|x|_\infty = |x|$, the usual absolute value. Show that

$$\prod_{p \leq \infty} |x|_p = 1$$

where the product ranges over all primes $p \in \mathbb{Z}$ as well as $p = \infty$.

Exercise 3.3. Let C be a hyperelliptic curve given by the Weierstrass equation

$$Y^2 = f(X, Z) = \prod_{i=1}^{2g+2} (X - \alpha_i Z)$$

in $\mathbb{P}(1, g+1, 1)$, where the $\alpha_i \in \mathbb{C}$ are distinct.

(a) Let U_0 and U_2 be the affine open subsets where $X \neq 0$ and $Z \neq 0$, respectively. Write $C \cap U_0$ and $C \cap U_2$ as affine curves given by (affine) Weierstrass equations.

(b) Determine the relationship between your local affine coordinates on U_0 and U_2. E.g., if x, y are affine coordinates on U_0 and z, w are affine coordinates on U_2, express z and w as rational functions in x and y, valid on $U_0 \cap U_2$.

(c) Compare our definition of a hyperelliptic curve with the one given in Miranda on p. 60. Do you see how they are equivalent?

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