BELYI MAPS AND DESSINS D'ENFANTS
HOMEWORK #2

Exercise 2.1. Show that the map

$$\begin{aligned} f : \mathfrak{H} &\to \mathfrak{D} \\ z &\mapsto \frac{z-i}{z+i} \end{aligned}$$

is an isomorphism of Riemann surfaces. What is its inverse?

Exercise 2.2. Show that the map

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{S}^2 \\ [z_0 : z_1] &\mapsto \frac{1}{|z_0|^2 + |z_1|^2}\left(2\operatorname{Re}(z_0\overline{z_1}), 2\operatorname{Im}(z_0\overline{z_1}), |z_0|^2 - |z_1|^2\right) \end{aligned}$$

is an isomorphism of Riemann surfaces.

Exercise 2.3. For a given $n \in \mathbb{Z}_{\geq 1}$, show that the map

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^1 \\ [X : Y] &\mapsto [X^n : Y^n] \end{aligned}$$

is a morphism of Riemann surfaces.

Exercise 2.4. Let E be the elliptic curve given by $Y^2Z = X^3 + Z^3$. Define the map

$$\begin{aligned} f : E &\to \mathbb{P}^1 \\ [X : Y : Z] &\mapsto [X : Z]\,. \end{aligned}$$

(a) Note that this expression for f does not work for the point $[0 : 1 : 0]$. How should $f([0 : 1 : 0])$ be defined in order to get a continuous map? (*Hint:* Write down equations for E and f on the affine open set where $Y \neq 0$. Use this affine equation for E to give a valid expression for f on this neighborhood.)

(b) Using your extended definition of f from the previous part, show that f is a morphism of Riemann surfaces.

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