BELYI MAPS AND DESSINS D'ENFANTS
HOMEWORK #1

Exercise 1.1. Let
$$X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$$
be the cone in $\mathbb{R}^3$ equipped with the subspace topology. Show that X cannot be given the structure of a topological surface.

Exercise 1.2. Fix $\omega_1, \omega_2 \in \mathbb{C}$ that are $\mathbb{R}$-linearly independent, and let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be the lattice they span. Let X be the quotient $\mathbb{C}/\Lambda$, a complex torus. Show that the collection of charts defined in lecture form a holomorphic atlas. (All that is left to check is that the transition functions are holomorphic.)

Exercise 1.3.

(a) Let $z = a + bi \in \mathbb{C}$. Considering $\mathbb{C}$ as an $\mathbb{R}$-vector space, then the left-multiplication map
$$\begin{aligned} \lambda_z : \mathbb{C} &\to \mathbb{C} \\ w &\mapsto zw \end{aligned}$$
is an $\mathbb{R}$-linear map. Determine the matrix of λ_z with respect to the basis $\{1, i\}$.

(b) Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function given by component functions $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$, so $h(x, y) = (h_1(x, y), h_2(x, y))$. Recall that its derivative is given by the Jacobian matrix
$$\begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \end{pmatrix}$$
Let $U \subseteq \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be holomorphic. Then $f(z) = u(x, y) + iv(x, y)$ is differentiable considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$, and its Jacobian matrix at z_0 must be the matrix of left multiplication by the complex number $f'(z_0)$.

Recover the Cauchy-Riemann equations by equating the Jacobian matrix and the matrix representation of $a + bi$ you found in the previous part.

Date: February 24, 2021.