1. (3 points)

   Prove that if V is a 1-dimensional vector space and $T \in \mathcal{L}(V)$, then there exists a scalar $\lambda \in \mathbb{F}$ such that $T(v) = \lambda(v)$ for all $v \in V$.

   Proof.

   We first pick a basis $\{v_i\}$ for V. *Since* V *is 1-dimensional*, the basis consists of a single vector $v_0 \in V$. Thus for any $v \in V$, there exists a scalar $c \in \mathbb{F}$ such that $v = cv_0$. In particular, since $T(v) \in V$, we define $\lambda \in \mathbb{F}$ to be the scalar such that

   $$T(v_0) = \lambda v_0.$$

   Now we compute $T(v)$. Because of linearity, we have

   $$T(v) = T(cv_0) = cT(v_0) = c\lambda v_0 = \lambda v.$$

   □

   Remark: Many people did not write why a basis of V consists of a single vector v_0. This is because the dimension of V is 1, and this should be mentioned: this problem is not true anymore when $\dim V \geq 2$.

   Another common mistake that people make is that they write something like

   $$\lambda = \frac{T(v)}{v}.$$

   $v \in V$ is not a scalar but a vector, and *you can't divide by a vector*!

   Finally, you can't pick $1 \in V$, this element does not exist until you *pick* an isomorphism

   $$V \cong \mathbb{F},$$

   and *this is a choice*! Indeed, $\times 2 : \mathbb{R} \to \mathbb{R}$ is also an isomorphism.

2. (8 points)

   a. Prove that there exists a linear map $T : \mathbb{F}^2 \to \mathbb{F}^3$ such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}, \quad T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix}.$$

   b. What is $T\begin{pmatrix}x\\y\end{pmatrix}$ for any $x, y \in \mathbb{F}$?

   c. Is T one-to-one?

Proof.
(a) Since $\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}$ is an invertible matrix, it follows that $\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\}$ is a basis for $\mathbb{F}^2$. By the linear extension theorem, such a linear map T exists!

(b) Since the conditions above show us that

$$T\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix} = \begin{pmatrix}1 & 1\\0 & -1\\2 & 4\end{pmatrix},$$

it follows that

$$\begin{aligned} T &= \begin{pmatrix}1 & 1\\0 & -1\\2 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}^{-1} \\ &= \begin{pmatrix}1 & 1\\0 & -1\\2 & 4\end{pmatrix}\begin{pmatrix}3 & -2\\-1 & 1\end{pmatrix} \\ &= \begin{pmatrix}2 & -1\\1 & -1\\2 & 0\end{pmatrix}. \end{aligned}$$

And so

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2 & -1\\1 & -1\\2 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x-y\\x-y\\2x\end{pmatrix}.$$

(c) Since we determined T in part (b), we just check that

$$\begin{pmatrix}2x-y\\x-y\\2x\end{pmatrix} = 0 \iff x = y = 0.$$

So T is one-to-one! □

3. (8 points)

   a. Let $U \coloneqq \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Find a basis of U.
   b. Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   c. Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Proof.
(a) First, we can check that

$$\{1, x-6, (x-6)^2, (x-6)^3, (x-6)^4\}$$

is a basis for $\mathcal{P}_4(\mathbb{R})$. Thus, we can write any $p \in \mathcal{P}_4(\mathbb{R})$ as

$$p = a_0 + a_1(x-6) + \cdots + a_4(x-6)^4$$

for some $a_0, \cdots, a_4 \in \mathbb{R}$. Then we see that

$$p''(6) = 0 \iff a_2 \neq 0.$$

So we can see that

$$\{1, x-6, (x-6)^3, (x-6)^4\}$$

is a basis for U. Check that the above is linearly independent and spans U.

(b) As noted in the proof above, we can extend the basis above to

$$\{1, x-6, (x-6)^2, (x-6)^3, (x-6)^4\}$$

which is a basis for $\mathcal{P}_4(\mathbb{R})$.

(c) From (b), it follows that $W \coloneqq \langle (x-6)^2 \rangle \subset \mathcal{P}_4(\mathbb{R})$ does it. □

4. (8 points)
Suppose $v_1, \cdots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \cdots, z_m) \coloneqq z_1 v_1 + \cdots + z_m v_m.$$

a. What property of T corresponds to $v_1, \cdots, v_m$ spanning V?
b. What property of T corresponds to $v_1, \cdots, v_m$ being linearly independent?

Give proofs of your claims in both cases.

Proof.
(a) **T surjective** $\iff$ $v_1, \cdots, v_m$ **spanning** V.

The equation to check here is that for any $v \in V$,

$$v = T(a_1, \cdots, a_m) \iff v = a_1 v_1 + \cdots + a_m v_m.$$

Thus $v \in V$ is in the image of T iff v is in the span of $v_1, \cdots, v_m$.

(b) **T injective** $\iff$ $v_1, \cdots, v_m$ **linearly independent.**

The equation to check here is that

$$T(a_1, \cdots, a_m) = 0 \iff a_1 v_1 + \cdots + a_m v_m = 0.$$

Thus $(a_1, \cdots, a_m) \in \mathbb{F}^m$ is in the kernel of T iff $a_1 v_1 + \cdots + a_m v_m = 0$, and such a nonzero vector exists iff $v_1, \cdots, v_m$ is not linearly independent!
$\square$

5. (10 points)
Suppose $v_1, \cdots, v_m$ is a list of vectors in V. For $k \in \{1, \cdots, m\}$, let

$$w_k \coloneqq v_1 + \cdots + v_k.$$

a. Show that $\mathrm{span}(v_1, \cdots, v_m) = \mathrm{span}(w_1, \cdots, w_m)$.
b. Show that $\{v_1, \cdots, v_m\}$ is linearly independent iff $\{w_1, \cdots, w_m\}$ is linearly independent.
c. Show that $v_1, \cdots, v_m$ is a basis of V iff $w_1, \cdots, w_m$ is a basis of V.

Proof.

(a) We can check the following two equations:

$$a_1 v_1 + \cdots + a_m v_m = a_1 w_1 + (a_2 - a_1) w_2 + \cdots + (a_m - a_{m-1}) w_m,$$
$$b_1 w_1 + \cdots + b_m w_m = (b_1 + \cdots + b_m) v_1 + (b_2 + \cdots + b_m) v_2 + \cdots + b_m v_m.$$

This shows that $\mathrm{span}(v_1, \cdots, v_m) = \mathrm{span}(w_1, \cdots, w_m)$.

(b) Check that the coefficients of the above equation satisfy

$$a_1 = a_2 - a_1 = \cdots = a_m - a_{m-1} = 0 \iff a_1 = \cdots = a_m = 0,$$
$$b_1 + \cdots + b_m = \cdots = b_m = 0 \iff b_1 = \cdots = b_m = 0.$$

(c) Since basis $\iff$ span + linearly independent, follows from (a), (b). $\square$