18.700 PROBLEM SET 3 SOLUTIONS

Due Wednesday, September 25 at 11:59 pm on Canvas

Collaborated with: no one
Sources used: none

Problem 1. (2A #5) (5 points)

(a) Find a number $t \in \mathbb{R}$ such that

$$(3,1,4),\ (2,-3,5),\ (5,9,t)$$

is linearly dependent in $\mathbb{R}^3$.

(b) For the value of t found in the previous part, determine the smallest $k \in \{1,2,3\}$ such that $v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$. Express the v_k as a linear combination of $v_1,\dots,v_{k-1}$.

Solution. (a) Suppose that

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = c_1\begin{pmatrix}3\\1\\4\end{pmatrix} + c_2\begin{pmatrix}2\\3\\-5\end{pmatrix} + c_3\begin{pmatrix}5\\9\\t\end{pmatrix} = \begin{pmatrix}3&2&5\\1&3&9\\4&-5&t\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}.$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Row reducing the corresponding augmented matrix, we find

$$\left(\begin{array}{ccc|c}3&2&5&0\\1&-3&9&0\\4&5&t&0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1&-3&9&0\\3&2&5&0\\4&5&t&0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1&-3&9&0\\0&11&-22&0\\0&17&t-36&0\end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c}1&-3&9&0\\0&1&-2&0\\0&17&t-36&0\end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c}1&-3&9&0\\0&1&-2&0\\0&0&t-2&0\end{array}\right).$$

If $t = 2$, then the last row becomes all zeroes, and the echelon form has only 2 pivots, in which case the system has infinitely many solutions. Infinitely many of these don't have c_1, c_2, c_3 all zero, hence the vectors are linearly dependent.

(b) Since v_2 is visibly not a scalar multiple of v_1, then $v_2 \notin \operatorname{span}(v_1)$. Thus $k = 3$ is the smallest such value. Putting the above matrix in RREF, we find

$$\left(\begin{array}{ccc|c}1&-3&9&0\\0&1&-2&0\\0&0&0&0\end{array}\right) \rightsquigarrow \begin{pmatrix}1&0&3&0\\0&1&-2&0\\0&0&0&0\end{pmatrix}.$$

The corresponding linear system is $c_1 + 3c_3 = 0$ and $c_2 - 2c_3 = 0$. Solving these for c_1 and c_2, we have

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}-3s\\2s\\s\end{pmatrix} = s\begin{pmatrix}-3\\2\\1\end{pmatrix}.$$

Taking $s = 1$ we find the linear relation $-3v_1 + 2v_2 + v_3 = 0$, and solving for v_3 yields

$$v_3 = 3v_1 - 2v_2 \in \operatorname{span}(v_1, v_2).$$

Problem 2. (2A #7) (5 points)

(a) Show that if we consider $\mathbb{C}$ as vector space over $\mathbb{R}$, the list $1+i, 1-i$ is linearly independent.

(b) Show that if we consider $\mathbb{C}$ as vector space over $\mathbb{C}$, the list $1+i, 1-i$ is linearly dependent.

Solution. (a) Suppose

$$0 = a_1(1+i) + a_2(1-i) = (a_1 + a_2) + (a_1 - a_2)i$$

for some $a_1, a_2 \in \mathbb{R}$. Since a complex number is 0 iff its real and imaginary parts are 0, we have

$$\begin{aligned} 0 &= a_1 + a_2 \\ 0 &= a_1 - a_2 . \end{aligned}$$

Adding these two equations together we find $2a_1 = 0$, hence $a_1 = 0$, and substituting this back into the first equation yields $a_2 = 0$. Thus the vectors are linearly independent.

(b) Considering $\mathbb{C}$ as a vector space over itself means that we can use complex numbers as coefficients when forming linear combinations. Since

$$(1-i)(1+i) - (1+i)(1-i) = 0$$

then the vectors are linearly dependent.

Problem 3. (2A #13) (6 points) Suppose that $v_1, \ldots, v_m \in V$ are linearly independent and $w \in V$. Show that $v_1, \ldots, v_m, w$ are linearly independent iff $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

Solution. We instead show that $v_1, \ldots, v_m, w$ are linearly dependent iff $w \in \operatorname{span}(v_1, \ldots, v_m)$. (This is equivalent to the problem statement because $P \iff Q$ is equivalent to the contrapositive $\neg Q \iff \neg P$.)

($\Rightarrow$): Assume that $v_1, \ldots, v_m, w$ are linearly dependent. Then there exist $a_1, \ldots, a_m, b \in \mathbb{F}$, not all 0, such that

$$0 = a_1 v_1 + \cdots + a_m v_m + bw .$$

We claim that $b \neq 0$. Suppose for contradiction that $b = 0$. Then

$$0 = a_1 v_1 + \cdots + a_m v_m .$$

Since $v_1, \ldots, v_m$ are linearly independent, then $a_1 = \cdots = a_m = 0$. But then all the coefficients are 0, contrary to our assumption. Thus $b \neq 0$. Then

$$w = \frac{1}{b}(-a_1 v_1 - \cdots - a_m v_m) \in \operatorname{span}(v_1, \ldots, v_m).$$

($\Leftarrow$): Suppose $w \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$w = a_1 v_1 + \cdots + a_m v_m .$$

Subtracting w, then

$$0 = a_1 v_1 + \cdots + a_m v_m - w .$$

Since the coefficient of w in the above equation is $-1 \neq 0$, then this is a nontrivial linear relation. Thus $v_1, \ldots, v_m, w$ are linearly dependent.

Problem 4. (1C #24) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

- f is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- f is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Let

$$V_e := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is even}\}, \text{ and}$$
$$V_o := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is odd}\}.$$

(a) Show that V_e and V_o are subspaces of $\mathbb{R}^{\mathbb{R}}$.
(b) Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Solution. (a) We apply the Subspace Criterion to V_e and V_o. Suppose $x \in \mathbb{R}$.

(i) We have

$$0(-x) = 0 = 0(x).$$

Thus the zero function is in V_e.

(ii) Given $f, g \in V_e$, then

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

so $f + g \in V_e$.

(iii) Given $c \in \mathbb{F}$ and $f \in V_e$, then

$$(cf)(-x) = cf(-x) = cf(x) = (cf)(x).$$

Thus $cf \in V_e$.

(i) We have

$$0(-x) = 0 = -0 = -0(x).$$

Thus the zero function is in V_o.

(ii) Given $f, g \in V_o$, then

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x),$$

so $f + g \in V_o$.

(iii) Given $c \in \mathbb{F}$ and $f \in V_o$, then

$$(cf)(-x) = cf(-x) = c(-f(x)) = -cf(x) = -(cf)(x).$$

Thus $cf \in V_e$.

(b) Given $f \in \mathbb{R}^{\mathbb{R}}$, let

$$f_e(x) = \frac{f(x) + f(-x)}{2}$$
$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Then $f = f_e + f_o$. Given $x \in \mathbb{R}$, we have

$$f_e(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = f_e(x)$$
$$f_o(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x),$$

so $f_e \in V_e$ and $f_o \in V_o$. Thus $V_e + V_o = \mathbb{R}^{\mathbb{R}}$.

Suppose $f \in V_e \cap V_o$. Given $x \in \mathbb{R}$, then $f(-x) = f(x)$ since f is even, and $f(-x) = -f(x)$ since f is odd. Then

$$f(x) = f(-x)$$
$$f(x) = -f(-x)$$

and adding these two equations together yields $2f(x) = 0$. Thus $f(x) = 0$ for all $x \in \mathbb{R}$, hence $f = 0$, the zero function. Thus $V_e \cap V_o = \{0\}$. By a previous result (Axler 1.46), then $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Problem 5. (11 points) Consider the following subspaces of $\mathbb{F}^3$:

$$V_1 := \{(x, y, 0) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}$$
$$V_2 := \{(x, 0, z) \in \mathbb{F}^3 \mid x, z \in \mathbb{F}\}$$
$$V_3 := \{(0, y, z) \in \mathbb{F}^3 \mid y, z \in \mathbb{F}\}.$$

(a) Compute $V_1 \cap V_2 \cap V_3$.
(b) Is the sum $V_1 + V_2 + V_3$ direct? Why or why not?
(c) Prove the following generalized criterion for a sum to be direct. Let V be a vector space and $V_1, \ldots, V_m$ be subspaces of V. Then the sum $V_1 + \cdots + V_m$ is direct iff

$$V_j \cap \left(\sum_{i \neq j} V_i\right) = V_j \cap \left(V_1 + \cdots V_{j-1} + V_{j+1} + \cdots + V_m\right) = \{0\}$$

for each $j = 1, \ldots, m$.

Solution.

(a) Suppose $v := (a, b, c) \in V_1 \cap V_2 \cap V_3$. Since $v \in V_1$, then

$$(a, b, c) = (x, y, 0)$$

for some $x, y \in \mathbb{F}$. Thus $c = 0$. Since $v \in V_2$, then

$$(a, b, c) = (u, 0, w)$$

for some $u, w \in \mathbb{F}$. Thus $b = 0$. Since $v \in V_3$, then

$$(a, b, c) = (0, s, t)$$

for some $s, t \in \mathbb{F}$. Thus $a = 0$. Therefore $(a, b, c) = (0, 0, 0)$, so $V_1 \cap V_2 \cap V_3 = \{0\}$.

(b) No, the sum is not direct. Observe that

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$
$$(0, 0, 0) = (1, 1, 0) + (-1, 0, 1) + (0, -1, -1).$$

These are two distinct representations of $(0,0,0)$ as a sum of elements in V_1, V_2, V_3, violating the uniqueness condition in the definition of a direct sum.

(c) $(\Rightarrow)$: Assume the sum is direct. Fix $j \in \{1, \ldots, m\}$ and suppose $v \in V_j \cap \left(\sum_{i \neq j} V_i \right)$.

Since $v \in \left(\sum_{i \neq j} V_i \right)$, then there exist $v_1 \in V_1, \ldots, v_{j-1} \in V_{j-1}, v_{j+1} \in V_{j+1}, \ldots, v_m \in V_m$ such that

$$v = v_1 + \cdots + v_{j-1} + 0 + v_{j+1} + \cdots + v_m \,. \tag{1}$$

But we also have

$$v = 0 + \cdots + 0 + v + 0 + \cdots + 0 \,. \tag{2}$$

Since the sum $\bigoplus_{i=1}^{m} V_i$ is direct, then there is a unique way of expressing v as a sum of elements of $V_1, \ldots, V_m$. Thus these two expressions must actually be the same. Equating the V_j term in (1) and (2), we have $v = 0$. Thus $V_j \cap \left(\sum_{i \neq j} V_i \right) = \{0\}$.

$(\Leftarrow)$: Assume $V_j \cap \left(\sum_{i \neq j} V_i \right) = \{0\}$ for all $j \in \{1, \ldots, m\}$. By a previous result (Axler 1.45), it suffices to show that the only way 0 can be written as $0 = v_1 + \cdots + v_m$ with $v_i \in V_i$ for $i = 1, \ldots, m$ is if $v_i = 0$ for all $i = 1, \ldots, m$.

Suppose

$$0 = v_1 + \cdots + v_j + \cdots + v_m$$

for some $v_i \in V_i$, $i = 1, \ldots, m$. Then for each $j = 1, \ldots, m$, we have

$$V_j \ni v_j = -v_1 - \cdots - v_{j-1} - v_{j+1} - \cdots - v_m \in \sum_{i \neq j} V_i \,.$$

Then

$$v_j \in V_j \cap \left(\sum_{i \neq j} V_i \right) = \{0\}$$

so $v_j = 0$. Thus $v_j = 0$ for all j, as desired. Therefore the sum is direct.