1. (6 points)

   a. Suppose a 3×5 coefficient matrix for a linear system has 3 pivot columns. Is the system consistent? Why or why not?

   b. Suppose a linear system has a 3×5 augmented matrix whose fifth column is a pivot column. Is the system consistent? Why or why not?

   c. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system must be consistent.

Proof.
(a) **The system is consistent**. Suppose we have a 3×6 augmented matrix for a linear system as above. Then if we reduce it into echelon form, the last column cannot be a pivot column, since the first five columns form the coefficient matrix, which already has three pivot columns. Thus, the leading nonzero entry already appears before the last column!

(b) **This system is inconsistent**. Since the last column is a pivot column, there is a row of the form $[0000b]$ in the augmented matrix, for some $b \neq 0$. As a linear equation, this is of the form

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = b \neq 0,$$

which cannot admit a solution.

(c) We need to show that this linear system has a solution for every constant vector, i.e. for any choice of the last column in the augmented matrix.

First, we look at the row-reduced echelon form of the augmented matrix. This means that all the entries of the pivot positions are 1. Now set the *free variables* to be zero. These correspond to the columns that do not contain a pivot position. Then since the fixed variables have coefficient 1, a solution for this linear system is determined by the last column. □

Remark: For (a), many people wrote that since there are free variables, the linear system is consistent. However, the definition of consistency is when every linear system of equations has a solution: it has nothing to do with free variables. In fact, even if you have many free variables, the equation still may not have any solutions.

Also, some students argued by an example where the three pivot columns are in the first, second, and third column: this is just a special case, so the proof is not done if you just argue for that case.

2. (5 points)
Suppose $a \in \mathbb{F}$, $v \in V$ such that $av = 0$. Prove that $a = 0$ or $v = 0$.

Proof.
Suppose $a \neq 0$. Then by the field axioms, there exists a multiplicative inverse $a^{-1} \in \mathbb{F}$. Now by the **associativity of scalar multiplication**,

$$v = (1_{\mathbb{F}}v) = (a^{-1}a)v = a^{-1}(av) = a^{-1}0 = 0.$$

The first equality holds because of the multiplicative identity axiom, the second line holds because a^{-1} is a multiplicative inverse of a, the third equality holds because of associativity of scalar multiplication. □

Here I want to make a remark. Some of the students assumed that you can write v entrywise as $(v_1, \cdots, v_n)$ and just concluded that $av_i = 0$ implies either $a = 0$ or $v_i = 0$. This is again true because of the **existence of a multiplicative inverse for nonzero elements in a field**.

Also, many students just deduced $v = 0$ from $av = 0$ and $a \neq 0$. However, to "divide by a", you need to first find a multiplicative inverse of a and then multiply it to the equation. Thus, you need to mention why such a multiplicative inverse exists.

Finally the equation

$$v = (a^{-1}a)v = a^{-1}(av) = 0$$

is not immediate: it uses many of the axioms in the definition of a vector space.

Although I did not deduct points for omitting the bold phrases, on the exam I want you to mention them.

3. (4 points)
Let W be the union of the first and third quadrants in $\mathbb{R}^2$, i.e.,

$$W \coloneqq \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}.$$

   a. Show that $cw \in W$ for all $c \in \mathbb{R}$ and all $w \in W$.
   b. Is W a vector space? Why or why not?

Proof.
(a) First if $c = 0$, clearly $cw = 0$ is an element of W. So now we assume $c \neq 0$. For $w = (x, y)$, we have

$$cw = c(x, y) = (cx, cy),$$

and since $(cx)(cy) = c^2xy \geq 0$ iff $xy \geq 0$, we conclude $cw \in W$.

(b) $(-1, 0), (0, 1) \in W$, but $(-1, 0) + (0, 1) = (-1, 1) \notin W$.
Thus, the **closure under additivity** fails, meaning that the **addition on** W **is not well-defined**. $\square$

Remark: one needs to mention the precise reason why W is not a vector space: it is because the addition inherited from W is not a well-defined operation on W.

4. (3 points)
Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{-\infty, \infty\}$ as follows.
The sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$\begin{aligned}
&t + \infty = \infty + t = \infty + \infty = \infty, \\
&t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) = -\infty, \\
&\infty + (-\infty) = (-\infty) + \infty = 0.
\end{aligned}$$

The product is defined as

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0. \end{cases}$$

and likewise for $t(-\infty)$.
With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{-\infty, \infty\}$ a vector space over $\mathbb{R}$? Explain.

Proof.
We claim that **associativity of addition** fails.

$$\begin{aligned}
&\infty + (\infty + (-\infty)) = \infty + 0 = \infty, \\
&(\infty + \infty) + (-\infty) = \infty + (-\infty) = 0.
\end{aligned}$$

Since they differ, we have shown that addition is not associative! □

5. (6 points)
Let $U = \{0\}$ and define $0 + 0 = 0$ and $\lambda \cdot 0 = 0$ for all $\lambda \in \mathbb{F}$. Prove that U is a vector space by verifying each of the vector space axioms.

Proof.
The book of Axler, Definition 1.20 has 6 axioms listed for the definition of a vector space. They are:

i. **commutativity of addition**,
ii. **associativity of addition and scalar multiplication**,
iii. **additive identity**,
iv. **additive inverse**,
v. **multiplicative identity**,
vi. **distributivity**.

Checking them is an easy exercise. □