Problem Set #1 Solutions

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Problem 1. Do the planes $x_1 + 2x_2 + x_3 = 4, x_2 - x_3 = 1$ and $x_1 + 3x_2 = 0$ have at least one common point of intersection?

Solution. It is sufficient to show whether the following linear system has at least one solution:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right).$$

To do this, we can row-reduce the matrix as follows

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array}\right) \xrightarrow{R_3 \leftarrow R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right).$$

We see that the last column contains a pivot, so the system has no solution. Hence the planes have no common point of intersection. □

Problem 2. Give an example of a 3×3 matrix A with real entries whose reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

and such that every entry of A is a nonzero integer.

Solution. Consider the following matrix, with nonzero, integer values:

$$\begin{pmatrix} 6 & 3 & 4 \\ 6 & 6 & 5 \\ 6 & 3 & 4 \end{pmatrix}.$$

If we row reduce it as follows

$$\begin{pmatrix} 6 & 3 & 4 \\ 6 & 6 & 5 \\ 6 & 3 & 4 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} 6 & 3 & 4 \\ 0 & 3 & 1 \\ 6 & 3 & 4 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{pmatrix} 6 & 3 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow (1/6) R_1} \begin{pmatrix} 1 & 1/2 & 2/3 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow (1/3) R_2} \begin{pmatrix} 1 & 1/2 & 2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftarrow R_2 - R_1} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$$

we get the desired reduced row-echelon form. Note that there are infinitely many matrices which have this reduced row-echelon form and the above is just one example. □

Problem 3. Fix $a, b, c, d \in \mathbb{F}$ and consider the matrix

$$A := \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}.$$

Assuming $a \neq 0$ and $c \neq 0$, compute the reduced row echelon form of A. (Hint: You will have to deal with two cases, depending on whether some quantity is zero or not.)

Solution. We row reduce

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftarrow (1/a)R_1} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - cR_1} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & d - bc/a & -c/a & 1 \end{pmatrix}.$$

Now, suppose $ad - bc = 0$. Then, since $a \neq 0, c \neq 0, -c/a$ is the pivot of the second row, so we continue row-reducing as follows:

$$\xrightarrow{R_2 \leftarrow (-a/c)R_2} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & 0 & 1 & -a/c \end{pmatrix} \xrightarrow{R_1 \leftarrow R_1 - (1/a)R_2} \begin{pmatrix} 1 & b/a & 0 & 1/c \\ 0 & 0 & 1 & -a/c \end{pmatrix}.$$

Otherwise, $ad - bc \neq 0$. Then $d - bc/a$ is the pivot of the second row, so we row-reduce as follows:

$$\xrightarrow{R_2 \leftarrow (a/(ad-bc))R_2} \begin{pmatrix} 1 & b/a & 1/a & 0 \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{pmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - (1/a)R_2} \begin{pmatrix} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{pmatrix}.$$

□

Problem 4. For each of the following augmented matrices, determine the value(s) of h such that the corresponding linear system is consistent.

(a)

$$\left(\begin{array}{cc|c} 1 & h & -3 \\ -2 & 4 & 6 \end{array}\right)$$

(b)

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ -4 & h & 8 \end{array}\right)$$

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Solution. We need to determine whether the system of linear equations defined by the matrix has at least one solution. To do this, we row-reduce each matrix:

(a)

$$\left(\begin{array}{cc|c} 1 & h & -3 \\ -2 & 4 & 6 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & h & -3 \\ 0 & 4+2h & 0 \end{array}\right).$$

Observe that for any value of h the above matrix does not have a pivot in the last column, hence the system is consistent for all values of h.

(b)

$$\left(\begin{array}{cc|c} 1 & 3 & -2 \\ -4 & h & 8 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{array}\right).$$

Again, observe that for any value of h the above matrix does not have a pivot in the last column, hence the system is consistent for all values of h.

□

Problem 5. Let

$$S := \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} : t \in \mathbb{F} \right\}.$$

Give a linear system whose solution set is S.

Solution. Observe that a vector (x_1, x_2, x_3) lies in S, ie satisfies

$$x_1 = t + 2$$

$$x_2 = -2t + 3$$

$$x_3 = t$$

$\iff$ it is a solution to the linear system

$$\begin{array}{cccc} x_1 & & -x_3 & = 2 \\ & x_2 & +2x_3 & = 3 \end{array}$$

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Problem 6. (a) Suppose that $p(t) = a_0 + a_1 t + a_2 t^2$ is a quadratic polynomial with $a_0, a_1, a_2 \in \mathbb{F}$, whose graph passes through the points $(1, 12), (2, 15)$, and $(3, 16)$. Find the coefficients a_0, a_1, a_2 by solving the following linear system.

$$\begin{array}{llll} a_0 & +a_1(1) & +a_2(1)^2 & = 12 \\ a_0 & +a_1(2) & +a_2(2)^2 & = 15 \\ a_0 & +a_1(3) & +a_2(3)^2 & = 16 \end{array}$$

(b) Suppose that $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ is a polynomial of degree n with $a_0, a_1, \ldots, a_n \in \mathbb{F}$, whose graph passes through the points $(u_1, v_1), (u_2, v_2), \ldots, (u_{n+1}, v_{n+1})$. Determine a linear system that the coefficients $a_0, a_1, \ldots, a_n$ must satisfy, and write down its corresponding augmented matrix.

Solution. (a) We can represent this system of equations by the following augmented matrix, which we row-reduce to find the solution, if it exists:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 1 & 3 & 9 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1. \end{array}\right)$$

We see that $a_2 = -1, 1a_1 = 3 - 3a_2 = 3 + 3 = 6, a_0 = 12 - a_1 - a_2 = 7$.

(b) Observe that this polynomial passes through the given $n+1$ points iff the coefficients $a_0, a_1, ..., a_n$ satisfy the linear system

$$a_0 + a_1(u_1) + a_2(u_1)^2 + ... + a_n(u_1)^n = v_1$$

$$a_0 + a_1(u_2) + a_2(u_2)^2 + ... + a_n(u_2)^n = v_2$$

$$...$$

$$a_0 + a_1(u_{n+1}) + a_2(u_{n+1})^2 + ... + a_n(u_{n+1})^n = v_{n+1}$$

with the corresponding augmented matrix being

$$\left(\begin{array}{ccccc|c} 1 & u_1 & u_1^2 & ... & u_1^n & v_1 \\ 1 & u_2 & u_2^2 & ... & u_2^n & v_2 \\ ... & & & & & \\ 1 & u_{n+1} & u_{n+1}^2 & ... & u_{n+1}^n & v_{n+1}. \end{array}\right)$$

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