18.700 MIDTERM FINAL EXAM REVIEW

1. Final Exam Information

- The final exam will take place Monday, December 16th at 1:30 - 4:30pm in Room 6-120 (the usual classroom).
- The exam will be cumulative, but with a greater emphasis on topics covered since the second midterm exam.
- Mathematics is not a spectator sport! The best way to prepare for the exam is to do problems.
- I also recommend making yourself a brief outline of the course material. Write down the most important results from each chapter.

2. PRACTICE EXAM

Problem 1.

(a) Let V be a finite-dimensional vector space and suppose $v_1, \ldots, v_n$ is a basis for V. Define $\varphi_1, \ldots, \varphi_n \in \mathcal{L}(V, \mathbb{F})$ by

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases}$$

Show that $\varphi_1, \ldots, \varphi_n$ is a basis for $\mathcal{L}(V, \mathbb{F})$. (This is called the *dual basis to* $v_1, \ldots, v_n$.)

(b) Now suppose that V is a finite-dimensional inner product space. Assume that the basis $v_1, \ldots, v_n$ from the previous part is orthonormal. For each $i = 1, \ldots, n$, show that

$$\varphi_i(u) = \langle u, v_i \rangle$$

for all $u \in V$.

(c) Prove that

$$\ker(\varphi_i) = (\operatorname{span}(v_i))^\perp$$

for each $i = 1, \ldots, n$.

Problem 2. Let V be a finite-dimensional inner product space, and suppose $S, T \in \mathcal{L}(V)$.

(a) Prove that $\ker(T^*) = \operatorname{img}(T)^\perp$.

(b) Prove that $(ST)^* = T^* S^*$.

Problem 3. (Axler 8A #14, 17) Let V be a finite-dimensional vector space.

(a) Suppose $T \in \mathcal{L}(V)$ is nilpotent. Show that if $\lambda \in \mathbb{F}$ is an eigenvalue of T, then $\lambda = 0$.

(b) Suppose $T \in \mathcal{L}(V)$ is nilpotent and $T \neq 0$. Prove that T is not diagonalizable.

(c) Suppose that $T \in \mathcal{L}(V)$ is nilpotent and satisfies $T^m = 0$ with $m \in \mathbb{Z}_{>0}$. Prove that $1 - T$ is invertible, and

$$(1 - T)^{-1} = I + T + \cdots + T^{m-1}\,.$$

Problem 4. (Axler 9C #1, 3)

(a) Prove or give a counterexample: if $S, T \in \mathcal{L}(V)$, then $\det(S + T) = \det(S) + \det(T)$.

(b) Let V be a finite-dimensional vector space. Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I + T) = 1$.

Problem 5. (Axler 8C #6) Let $V := \mathcal{P}_4(\mathbb{R})$ and define $T \in \mathcal{L}(V)$ by $T(f) := f'$.

(a) Compute the eigenvalues of T together with their geometric multiplicities.

(b) Is T diagonalizable? Why or why not?

(c) Find a Jordan basis for T.

Problem 6. (Axler 5A #37) Let $n \in \mathbb{Z}_{\geq 0}$ and fix $A \in M_{n \times n}(\mathbb{F})$. Define $T \in \mathcal{L}(M_{n \times n}(\mathbb{F}))$ by

$$T(B) = AB$$

for all $B \in M_{n\times n}(\mathbb{F})$. Prove that the set of eigenvalues of T equals the set of eigenvalues of A. (*Hint*: Given an eigenvector v of A, consider the matrix B all of whose columns are v.)

Problem 7. Let

$$v_1 = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}.$$

(a) Apply the Gram-Schmidt procedure to v_1, v_2, v_3 to produce an orthonormal basis for $\mathbb{F}^3$.

(b) Let

$$A = \begin{pmatrix} 3 & 1 & -3 \\ 6 & 2 & -1 \\ 0 & 2 & 4 \end{pmatrix}.$$

Use your answer from the previous part to produce the QR decomposition of A. That is, find a unitary matrix Q and an upper triangular matrix R with positive diagonal entries such that $A = QR$.

Problem 8. Let V be a finite-dimensional inner product space.

(a) Suppose $Q \in \mathcal{L}(V)$ is unitary. Show that $\|Q(v)\| = \|v\|$ for all $v \in V$.

(b) Suppose that $Q_1, Q_2 \in \mathcal{L}(V)$ are unitary. Show that $Q_1 Q_2$ is unitary.

Problem 9. (Axler 7E #13)

(a) Let V be a finite-dimensional vector space. Suppose $T_1, T_2 \in \mathcal{L}(V)$ and $T_1 = P^{-1} T_2 P$ for some invertible $P \in \mathcal{L}(V)$. Show that T_1 and T_2 have the same eigenvalues.

(b) Now let V be a finite-dimensional inner product space. Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values iff there exist unitary operators $Q_1, Q_2 \in \mathcal{L}(V)$ such that

$$T_1 = Q_1 T_2 Q_2.$$

Problem 10. Give an example of a square matrix that is not diagonal, but is diagonalizable.