18.700 MIDTERM EXAM 1 REVIEW

1. Overview of Topics

(1) Systems of Linear Equations
  (a) Definition of a linear system
  (b) Elementary row operations
  (c) Row reduction and RREF
  (d) Pivots and free variables
  (e) 3 possibilities for number of solutions: none, one, or infinitely many
  (f) Parametric description of solution sets
  (g) Test for consistency using echelon form
(2) Vector Spaces
  (a) Definition of a vector space
  (b) Definition of a subspace
  (c) Subspace criterion
  (d) Sums and direct sums of subspaces
    (i) $V_1 + V_2$ is direct iff $V_1 \cap V_2 = \{0\}$.
  (e) Common examples of vector spaces: $\mathbb{F}^n, \mathbb{F}^\infty, \mathbb{F}^S, \mathcal{P}(\mathbb{F}), \mathcal{P}_m(\mathbb{F})$
(3) Finite-Dimensional Vector Spaces
  (a) Linear combinations
  (b) Span
    (i) $\text{span}(L)$ is the smallest subspace containing L. (Axler 2.6)
  (c) Linear (in)dependence
    (i) Linear Dependence Lemma (Axler 2.19)
    (ii) LI $\leq$ span (Axler 2.22)
  (d) Bases
    (i) A list of vectors is a basis iff it is linearly independent and spans.
    (ii) Reduction Lemma: Every spanning list can be reduced to a basis. (Axler 2.30)
    (iii) Extension Lemma: Every linearly independent list can be extended to basis. (Axler 2.32)
  (e) Dimension
    (i) Independence of Length Theorem: Any two bases of a finite-dimensional vector space have the same length. (Axler 2.34)
    (ii) U a subspace of $V \implies \dim(U) \leq \dim(V)$. (Axler 2.37)
    (iii) Dimension of a sum (Axler 2.43)
(4) Linear Maps
  (a) Definition of a linear map
    (i) Defining a linear map through its action on a basis. (Axler 3.4)
    (ii) $\mathcal{L}(V, W)$ is itself a vector space. (Axler 3.6)
  (b) Kernels aka null spaces

(c) Images aka ranges
(d) Injectivity and surjectivity
    (i) T is injective iff $\ker(T) = \{0\}$. (Axler 3.15)
    (ii) Rank-Nullity Theorem (Axler 3.21)

(5) Matrices
(a) The matrix of a linear map with respect to a choice of bases. (Axler 3.31)
(b) $[\lambda S + T] = \lambda[S] + [T]$, i.e., $[\cdot] : \mathcal{L}(V, W) \to \mathrm{M}_{m\times n}(\mathbb{F})$ is a linear map. (Axler 3.35, 3.38)
(c) Definiton of matrix multiplication (Axler 3.41)
(d) $[ST] = [S][T]$ (Axler 3.43)
(e) Interpretations of matrix multiplication in terms of rows and columns. (Axler 3.48, 3.51)
(f) Row rank = column rank.

(6) Invertibility and Isomorphisms
(a) Invertible linear maps aka isomorphisms
    (i) Invertible $\iff$ bijective. (Axler 3.63)
    (ii) If $\dim(V) = \dim(W)$, injective $\iff$ surjective. (Axler 3.65)
(b) Isomorphic vector spaces
    (i) $V \cong W \iff \dim(V) = \dim(W)$. (Axler 3.70)

2. Practice Problems

Problem 1. Let V and W be vector spaces, and let $T \in \mathcal{L}(V, W)$. Suppose that Z is a subspace of W. Show that

$$U := \{v \in V : T(v) \in Z\}$$

is a subspace of V.

Problem 2. Let $V = \mathcal{P}_3(\mathbb{R})$ and let $\mathcal{B} := (1, 2z, -2 + 4z^2, -12z + 8z^3)$. (These are the first four *Hermite polynomials*.)

(a) Show that $\mathcal{B}$ is basis of $\mathcal{P}_3(\mathbb{R})$.

(b) Let $p(z) = 7 - 12z - 8z^2 + 12z^3$. Find $[p]_{\mathcal{B}}$, the coordinate vector of p with respect to $\mathcal{B}$.

Problem 3. Let $V := \mathcal{P}_3(\mathbb{R})$ and let $T \in \mathcal{L}(V)$ be the linear map $T := \dfrac{d^2}{dx^2} + \dfrac{d}{dx}$, so

$$T(f) = f'' + f'.$$

Compute the matrix of T with respect to the standard monomial basis $1, x, x^2, x^3$.

Problem 4. (2A #8, 9)

(a) Suppose $v_1, v_2, v_3, v_4 \in V$ are linearly independent. Show that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

(b) Prove or give a counterexample: if $v_1, v_2, \ldots, v_m \in V$ are linearly independent, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

are also linearly independent.

Problem 5. (3A #13) Let V be a finite-dimensional vector space. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $T(u) = S(u)$ for all $u \in U$.

Problem 6. (3B #5, 6)

(a) Give an example of $T \in \mathcal{L}(\mathbb{R}^4)$ such that $\operatorname{img}(T) = \ker(T)$.

(b) Prove that there does not exist $T \in \mathcal{L}(\mathbb{R}^5)$ such that $\operatorname{img}(T) = \ker(T)$.

Problem 7. (3B #27) Let V be a vector space. Suppose $T \in \mathcal{L}(V)$ such that $T^2 = T$. Prove that $V = \ker(T) \oplus \operatorname{img}(T)$.

Problem 8. (3C #5) Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $\mathcal{B}$ of V and a basis $\mathcal{C}$ of W such that $A := {}_{\mathcal{C}}[T]_{\mathcal{B}}$ has the following form: $A_{ii} = 1$ for $i = 1, \ldots, \dim(\operatorname{img}(T))$ and $A_{ij} = 0$ otherwise.

Problem 9. (3C #10) Give an example of 2×2 matrices A and B such that $AB \neq BA$.

Problem 10. (3D #2) Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.