18.700 PROBLEM SET 9 (OPTIONAL)

This is an optional problem set that is not to be turned in, and will not be graded.

Collaborated with:
Sources used:

Problem 1. (8A #2) Suppose $T \in \mathcal{L}(V)$, $m \in \mathbb{Z}_{>0}$, and $v \in V$ such that $T^{m-1}(v) \neq 0$ but $T^m(v) = 0$. Prove that $v, T(v), T^2(v), \ldots, T^{m-1}(v)$ is linearly independent.

Problem 2. (8A #9) Suppose $T \in \mathcal{L}(V)$ and $m \in \mathbb{Z}_{\geq 0}$. Prove that

$$\ker(T^m) = \ker(T^{m+1}) \iff \operatorname{img}(T^m) = \operatorname{img}(T^{m+1})\,.$$

Problem 3. (8B #9) Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Prove that there exist $D, N \in \mathcal{L}(V)$ such that $T = D + N$, the operator D is diagonalizable, N is nilpotent, and $DN = ND$.

Problem 4. (8B #16) Let T be the operator on $\mathbb{C}^6$ defined by

$$T(z_1, z_2, z_3, z_4, z_5, z_6) = (0, z_1, z_2, 0, z_4, 0)\,.$$

Find $\operatorname{minpoly}(T)$ and $\operatorname{charpoly}(T)$.

Problem 5. (8C #5) Suppose $T \in \mathcal{L}(\mathbb{C}^2)$ is the operator defined by

$$T(w, z) = (-w - z, 9w + 5z)\,.$$

Find a Jordan basis for T.

Problem 6. (8D #5) Suppose V is an inner product space. Suppose $T \in \mathcal{L}(V)$ is a positive operator and $\operatorname{tr}(T) = 0$. Prove that $T = 0$.