18.700 PROBLEM SET 8

Due Wednesday, November 27 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Let V and W be nonzero finite-dimensional inner product spaces over $\mathbb{F}$.

Problem 1. (7A #4) (7 points) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that

$$U \text{ is } T\text{-invariant} \iff U^{\perp} \text{ is } T^*\text{-invariant.}$$

Problem 2. (7A #5) (6 points) Suppose $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Prove that

$$\|T(e_1)\|^2 + \cdots + \|T(e_n)\|^2 = \|T^*(f_1)\|^2 + \cdots + \|T^*(f_m)\|^2 .$$

Problem 3. (7B #1) (7 points) Let $\mathbb{F} = \mathbb{C}$ and suppose $T \in \mathcal{L}(V)$ is normal. Show that T is self-adjoint if and only if all the eigenvalues of T are real.

Problem 4. (7B #6) (6 points) Let $\mathbb{F} = \mathbb{C}$ and suppose $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$. (*Hint*: Use the previous exercise.)

Problem 5. (7C #3) (6 points) Let n be a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ be the operator whose matrix with respect to the standard basis consists of all 1s. Show that T is a positive operator.

Problem 6. (7E #3) (4 points) Give an example of $T \in \mathcal{L}(\mathbb{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are $5, 0$.