18.700 PROBLEM SET 7

Due Wednesday, November 6 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Let V be an inner product space.

Problem 1. (6A #9, 11) (10 points)

(a) Suppose $u, v \in V$, and $\|u\| = 1 = \|v\|$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

(b) Find vectors $u, v \in \mathbb{R}^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$, and $(1, 2) = u + v$.

Problem 2. (6A #16) (5 points) Given $x, y \in V$, we defined the angle between x and y to be

$$\angle(x, y) := \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right).$$

Explain why this definition makes sense using the Cauchy-Schwarz inequality.

Problem 3. (6B #1) (6 points) Suppose $e_1, \ldots, e_m$ is a list of vectors in V such that

$$\|a_1e_1 + \cdots + a_me_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbb{F}$. Show that $e_1, \ldots, e_m$ is an orthonormal list. (This exercise provides a converse to 6.24 in the textbook.)

Problem 4. (6B #7) (6 points) Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper-triangular matrix with respect to the basis $(1, 0, 0), (1, 1, 1), (1, 1, 2)$. Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Problem 5. (6C #3) (8 points) Suppose U is the subspace of $\mathbb{R}^4$ defined by

$$U := \operatorname{span}\left((1, 2, 3, -4), (-5, 4, 3, 2)\right).$$

Find an orthonormal basis of U and an orthonormal basis of $U^\perp$.

Problem 6. (6C #4) (10 points) Suppose $e_1, \ldots, e_n$ is a list of vectors in V with $\|e_k\| = 1$ for each $k = 1, \ldots, n$, and

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2$$

for all $v \in V$. Prove that $e_1, \ldots, e_n$ is an orthonormal basis of V. (This provides a converse to 6.30(b) in the textbook.)