18.700 PROBLEM SET 6

Due Wednesday, October 30 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Problem 1. (5B #21) (5 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of T has degree at most $1 + \dim(\operatorname{img}(T))$. (Observe that if $\dim(\operatorname{img}(T)) < \dim(V) - 1$, then this exercise gives a better upper bound for the degree of $\operatorname{minpoly}(T)$.) (*Hint*: Recall that $\operatorname{img}(T)$ is T-invariant.)

Problem 2. (5B #22) (6 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible iff $I \in \operatorname{span}(T, T^2, \ldots, T^{\dim(V)})$.

Problem 3. (5C #6) (5 points) Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that V has a k-dimensional T-invariant subspace for each $k = 1, \ldots, \dim(V)$.

Problem 4. (5C #11) (6 points) A square matrix is *lower-triangular* if all its entries above the diagonal are 0. Suppose $\mathbb{F} = \mathbb{C}$ and V is finite-dimensional. Prove that if $T \in \mathcal{L}(V)$, then there exists a basis of V with respect to which T has a lower-triangular matrix.

Problem 5. (5D #1) (7 points) Suppose V is a finite-dimensional $\mathbb{C}$-vector space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.
(b) Prove that if $T^4 = T$, then T is diagonalizable.
(c) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable. (Make sure to justify why your T is not diagonalizable.)

Problem 6. (5D #21) (12 points) The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by

$$F_0 = 0, \ F_1 = 1, \text{ and } F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 2.$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each $n \in \mathbb{Z}_{\geq 0}$.
(b) Find the eigenvalues of T.
(c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of T.
(d) Use the solution to (c) to compute $T^n(0, 1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for each $n \in \mathbb{Z}_{\geq 0}$. (The number $(1 + \sqrt{5})/2$ is called the *golden ratio*.)