18.700 PROBLEM SET 5

Due Wednesday, October 23 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Problem 1. (3D #18) (8 points) Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic $\mathbb{F}$-vector spaces.

Problem 2. (3D #6) (10 points) Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\ker(S) = \ker(T)$ if and only if there exists an invertible $P \in \mathcal{L}(W)$ such that $S = PT$.

Problem 3. (5A #20) (8 points) Define the left shift operator $L \in \mathcal{L}(\mathbb{F}^\infty)$ by

$$L(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

(a) Show that every element of $\mathbb{F}$ is an eigenvalue of L.
(b) Find all eigenvectors of L.

Problem 4. (5A #21) (8 points) Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.
(b) Prove that T and T^{-1} have the same eigenvectors.

Problem 5. (5B #11) (10 points) Suppose V is a 2-dimensional vector space, $T \in \mathcal{L}(V)$, and the matrix of T with respect to some basis of V is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that $T^2 - (a+d)T + (ad - bc)I = 0$.
(b) Show that the minimal polynomial of T is

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d, \\ z^2 - (a+d)z + (ad - bc) = 0 & \text{otherwise.} \end{cases}$$

Problem 6. (5A #30) (5 points) Suppose $T \in \mathcal{L}(V)$ and

$$(T - 2I)(T - 3I)(T - 4I) = 0.$$

Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.