18.700 PROBLEM SET 4

Due Wednesday, October 2 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Problem 1. (3A #7) (3 points) Prove that if V is a 1-dimensional vector space and $T \in \mathcal{L}(V)$, then there exists a scalar $\lambda \in \mathbb{F}$ such that $T(v) = \lambda v$ for all $v \in V$.

Problem 2. (8 points)

(a) Prove that there exists a linear map $T : \mathbb{F}^2 \to \mathbb{F}^3$ such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix} \qquad \text{and} \qquad T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix}.$$

(b) What is $T\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right)$ for any $x, y \in \mathbb{F}$?

(c) Is T one-to-one?

Problem 3. (2C #4) (8 points)

(a) Let $U := \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

Problem 4. (3B #3) (8 points) Suppose $v_1, \dots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \dots, z_m) := z_1 v_1 + \cdots + z_m v_m\,.$$

(a) What property of T corresponds to $v_1, \dots, v_m$ spanning V?

(b) What property of T corresponds to the list $v_1, \dots, v_m$ being linearly independent?

Give proofs of your claims in both cases. (Be sure to prove both directions of any "iff" statements.)

Problem 5. (2A #3, #14, 2B #9) (10 points) Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k := v_1 + \cdots + v_k\,.$$

(a) Show that $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

(b) Show that the list $v_1, \dots, v_m$ is linearly independent iff the list $w_1, \dots, w_m$ is linearly independent.

(c) Show that $v_1, \dots, v_m$ is a basis of V iff $w_1, \dots, w_m$ is a basis of V.

(*Hint:* It may be useful to find a formula for v_k in terms of $w_1, \dots, w_m$.)