18.700 PROBLEM SET 3

Due Wednesday, September 25 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Problem 1. (2A #5) (5 points)

(a) Find a number $t \in \mathbb{R}$ such that
$$(3,1,4),\ (2,-3,5),\ (5,9,t)$$
is linearly dependent in $\mathbb{R}^3$.

(b) For the value of t found in the previous part, determine the smallest $k \in \{1,2,3\}$ such that $v_k \in \operatorname{span}(v_1,\dots,v_{k-1})$. Express the v_k as a linear combination of $v_1,\dots,v_{k-1}$.

Problem 2. (2A #7) (5 points)

(a) Show that if we consider $\mathbb{C}$ as vector space over $\mathbb{R}$, the list $1+i, 1-i$ is linearly independent.

(b) Show that if we consider $\mathbb{C}$ as vector space over $\mathbb{C}$, the list $1+i, 1-i$ is linearly dependent.

Problem 3. (2A #13) (6 points) Suppose that $v_1,\dots,v_m \in V$ are linearly independent and $w \in V$. Show that $v_1,\dots,v_m,w$ are linearly independent iff $w \notin \operatorname{span}(v_1,\dots,v_m)$.

Problem 4. (1C #24) (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

- f is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- f is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Let
$$V_e := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is even}\}, \text{ and}$$
$$V_o := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is odd}\}.$$

(a) Show that V_e and V_o are subspaces of $\mathbb{R}^{\mathbb{R}}$.

(b) Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Problem 5. (11 points) Consider the following subspaces of $\mathbb{F}^3$:
$$V_1 := \{(x,y,0) \in \mathbb{F}^3 \mid x,y \in \mathbb{F}\}$$
$$V_2 := \{(x,0,z) \in \mathbb{F}^3 \mid x,z \in \mathbb{F}\}$$
$$V_3 := \{(0,y,z) \in \mathbb{F}^3 \mid y,z \in \mathbb{F}\}.$$

(a) Compute $V_1 \cap V_2 \cap V_3$.

(b) Is the sum $V_1 + V_2 + V_3$ direct? Why or why not?

(c) Prove the following generalized criterion for a sum to be direct. Let V be a vector space and $V_1, \dots, V_m$ be subspaces of V. Then the sum $V_1 + \cdots + V_m$ is direct iff

$$V_j \cap \left(\sum_{i \neq j} V_i \right) = V_j \cap \left(V_1 + \cdots V_{j-1} + V_{j+1} + \cdots + V_m \right) = \{0\}$$

for each $j = 1, \dots, m$.