18.700 PROBLEM SET 2

Due Wednesday, September 18 at 11:59 pm on Canvas

Collaborated with:
Sources used:

Note: When using one or more of the vector space axioms in your solutions, make sure to indicate which axioms are being used and how. E.g., "…where the second equality follows by associativity of scalar multiplication."

Problem 1. (6 points)

(a) Suppose a 3×5 *coefficient* matrix for a linear system has 3 pivot columns. Is the system consistent? Why or why not?
(b) Suppose a linear system has a 3×5 *augmented* matrix whose fifth column is a pivot column. Is the system consistent? Why or why not?
(c) Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system must be consistent.

Problem 2. (1B, #2) (5 points) Suppose $a \in \mathbb{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Problem 3. (4 points) Let W be the union of the first and third quadrants in $\mathbb{R}^2$, i.e.,

$$W := \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}.$$

(a) Show that $cw \in W$ for all $c \in \mathbb{R}$ and all $w \in W$.
(b) Is W a vector space? Why or why not?

Problem 4. (1B, #6) (3 points) Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as follows. The sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

Problem 5. (6 points) Let $U = \{0\}$ and define $0 + 0 = 0$ and $\lambda\, 0 = 0$ for all $\lambda \in \mathbb{F}$. Prove that U is a vector space by verifying each of the vector space axioms.