INNER PRODUCTS AND ORTHOGONALIZATION WORKSHEET

NOVEMBER 4, 2024

(1) Let $V = \mathcal{P}_2(\mathbb{R})$ and define an inner product on V by

$$\langle p, q \rangle := \int_{-1}^{1} p(x)q(x)\, dx\,.$$

Starting with the basis $1, x, x^2$ for V, compute an orthonormal basis using the Gram-Schmidt procedure.

(2) Let $V := \mathbb{R}^3$ and let

$$u_1 := \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, \; u_2 := \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \; v := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Compute the point of U that is closest to v, where $U = \operatorname{span}(u_1, u_2)$.

(3) Let V be an inner product space.

(a) Given $v \in V$ and $\lambda \in \mathbb{F}$, show that $\|\lambda v\| = |\lambda|\|v\|$. (*Hint*: Compute $\|\lambda v\|^2$ in terms of the inner product.)

(b) Suppose $u, v \in V$ and $u \perp v$. Prove the Pythagorean theorem:

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 .$$

(c) Prove the Parallelogram Identity:

$$\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

for all $u, v \in V$.