INNER PRODUCTS AND ORTHOGONALIZATION WORKSHEET

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(1) Let $V = \mathcal{P}_2(\mathbb{R})$ and define an inner product on V by

$$\langle p, q \rangle := \int_{-1}^{1} p(x)q(x)\, dx\,.$$

Starting with the basis $1, x, x^2$ for V, compute an orthonormal basis using the Gram-Schmidt procedure.

Solution. We begin by computing an orthogonal basis f_1, f_2, f_3 using the Gram-Schmidt procedure. Letting $f_1 := 1$, then

$$\|f_1\|^2 = \langle 1, 1 \rangle = \int_{-1}^{1} 1\, dx = x\Big|_{-1}^{1} = 2\,,$$

so

$$e_1 := \frac{1}{\|f_1\|} f_1 = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}\,.$$

Now

$$f_2 := x - \frac{\langle x, f_1 \rangle}{\|f_1\|^2} = x - \frac{1}{2}\int_{-1}^{1} x \cdot 1\, dx = x - \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = x - 0 = x$$

and

$$\|f_2\|^2 = \langle x, x \rangle = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}\,,$$

so

$$e_2 := \frac{1}{\|f_2\|} f_2 = \sqrt{\frac{3}{2}} x\,.$$

Finally,

$$\begin{aligned} f_3 &:= x^2 - \frac{\langle x^2, f_1 \rangle}{\|f_1\|^2} - \frac{\langle x^2, f_2 \rangle}{\|f_2\|^2} = x^2 - \left(\frac{1}{2}\int_{-1}^{1} x^2 \cdot 1\, dx\right) - \left(\frac{3}{2}\int_{-1}^{1} x^2 \cdot x\, dx\right) \\ &= x^2 - \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} - \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = x^2 - \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} \|f_3\|^2 &= \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx = \frac{x^5}{5} - \frac{2}{3}\frac{x^3}{3} + \frac{1}{9}x\Big|_{-1}^{1} \\ &= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}\,, \end{aligned}$$

so

$$e_3 := \frac{1}{\|f_3\|} f_3 = \frac{3\sqrt{5}}{2\sqrt{2}} \left(x^2 - \frac{1}{3} \right) = \frac{\sqrt{5}}{2\sqrt{2}} (3x^2 - 1)\,.$$

(2) Let $V := \mathbb{R}^3$ and let

$$u_1 := \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix},\ u_2 := \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},\ v := \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Compute the point of U that is closest to v, where $U = \operatorname{span}(u_1, u_2)$.

Solution. By a result in class, the projection $\operatorname{proj}_U(v)$ is the point in U closest to v, so it suffices to compute this projection. Note that u_1 and u_2 are already orthogonal, so it suffices to normalize them to produce an orthonormal basis of U. We have

$$\begin{aligned} \|u_1\| &= \sqrt{4 + 25 + 1} = \sqrt{30} \\ \|u_2\| &= \sqrt{4 + 1 + 1} = \sqrt{6} \end{aligned}$$

so

$$\begin{aligned} e_1 &:= \frac{1}{\|u_1\|} u_1 = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \\ e_2 &:= \frac{1}{\|u_2\|} u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \end{aligned}$$

is an orthonormal basis for U. Then

$$\begin{aligned} \operatorname{proj}_U(v) &= \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 = \left\langle v, \frac{1}{\|u_1\|} u_1 \right\rangle \frac{1}{\|u_1\|} u_1 + \left\langle v, \frac{1}{\|u_2\|} u_2 \right\rangle \frac{1}{\|u_2\|} u_2 \\ &= \frac{\langle v, u_1 \rangle}{\|u_1\|^2} u_1 + \frac{\langle v, u_2 \rangle}{\|u_2\|^2} u_2 \\ &= \frac{1 \cdot 2 + 2 \cdot 5 + 3 \cdot (-1)}{30} u_1 + \frac{1 \cdot (-2) + 2 \cdot 1 + 3 \cdot 1}{6} u_2 \\ &= \frac{9}{30} \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{3}{6} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}. \end{aligned}$$

(3) Let V be an inner product space.

(a) Given $v \in V$ and $\lambda \in \mathbb{F}$, show that $\|\lambda v\| = |\lambda|\|v\|$. (*Hint*: Compute $\|\lambda v\|^2$ in terms of the inner product.)

Solution. Since the inner product is linear in the first component and conjugate-linear in the second, then

$$\|\lambda v\|^2 = \langle \lambda v, \lambda v\rangle = \lambda\overline{\lambda}\langle v, v\rangle = |\lambda|^2\|v\|^2$$

Both sides are nonnegative, so taking square roots yields the desired result.

(b) Suppose $u, v \in V$ and $u \perp v$. Prove the Pythagorean theorem:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 .$$

Solution. Since u and v are orthogonal, then $\langle u, v\rangle = 0$, so

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u\rangle + \cancelto{0}{\langle u, v\rangle} + \langle v, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \cancelto{0}{\overline{\langle u, v\rangle}} + \|v\|^2 = \|u\|^2 + \|v\|^2 .\end{aligned}$$

(c) Prove the Parallelogram Identity:

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

for all $u, v \in V$.

Solution. Given $u, v \in V$, then

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2\end{aligned}$$

and

$$\begin{aligned}\|u-v\|^2 &= \langle u-v, u-v\rangle = \langle u, u\rangle + \langle u, -v\rangle + \langle -v, u\rangle + \langle -v, -v\rangle \\ &= \|u\|^2 - \langle u, v\rangle - \langle v, u\rangle + \|v\|^2 .\end{aligned}$$

Then

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \|u\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|v\|^2 \\ &\quad + \|u\|^2 - \langle u, v\rangle - \langle v, u\rangle + \|v\|^2 \\ &= 2\left(\|u\|^2 + \|v\|^2\right) .\end{aligned}$$