18.700 - LINEAR ALGEBRA, DAY 9
INVERTIBILITY AND ISOMORPHISMS

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CONTENTS

I. PRE-CLASS PLANNING

I.1. **Goals for lesson.**

(1) Students will learn the definition of matrix multiplication.
(2) Students will learn that the row rank = the column rank of a matrix.
(3) Students will learn the definition of invertibility and isomorphism.
(4) Students will learn that an n-dimensional vector space is isomorphic to $\mathbb{F}^n$.
(5) Students will learn that if $\dim(V) = n$ and $\dim(W) = m$, then $\mathcal{L}(V, W) \cong \mathrm{M}_{\mathrm{m}\times\mathrm{n}}(\mathbb{F})$.
(6) Students will learn that $[T(v)]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}[v]_{\mathcal{B}}$.
(7) Students will learn the change of basis formula.

I.2. **Methods of assessment.**

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. **Materials to bring.** (1) Laptop + adapter (2) Worksheets

(0:00)

II. Lesson Plan

Announcements: • Exam 1: Wednesday, October 9th in class. No pset this week; instead review packet. • TA office hours: Tuesday, Oct 8th, 7:00 - 9:00pm, 2-361

II.1. **Last time.**

- Defined the image of a linear map.
- Rank-Nullity Theorem: If $T : V \to W$ is linear, then $\dim(V) = \dim(\ker(T)) + \dim(\operatorname{img}(T))$.
- Defined coordinate vector $[v]_{\mathcal{B}} \in \mathbb{F}^n$ for $v \in V$.
- Defined the matrix $[T]$ of a linear map with respect to a choice of bases.

Remark 1. Linear vs affine. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + b$ is linear iff $b = 0$. (In general, these translates of linear maps are called *affine maps*.)

II.2. **Matrix multiplication.** Suppose that U, V, W are finite-dimensional vector spaces with bases

$$\begin{aligned}\mathcal{B} &:= (u_1, \ldots, u_p)\\ \mathcal{C} &:= (v_1, \ldots, v_n)\\ \mathcal{D} &:= (w_1, \ldots, w_m)\,.\end{aligned}$$

Suppose $T : U \to V$ and $S : V \to W$ are linear maps. We previously saw that the composition $ST : U \to W$ is linear. We now define matrix multiplication in such a way that

$$[ST] = [S][T]\,.$$

Let $A := [S]$ and $B := [T]$. Then for each $j = 1, \ldots, p$ we have

$$(ST)(u_j) = S\left(\sum_{k=1}^{n} B_{kj} v_k\right) = \sum_{k=1}^{n} B_{kj} S(v_k) = \sum_{k=1}^{n} B_{kj} \sum_{i=1}^{m} A_{ik} w_k = \sum_{i=1}^{m}\sum_{k=1}^{n} \left(A_{ik}B_{kj}\right) w_k\,.$$

Thus $[ST]$ is the $m \times p$ matrix whose i, j entry is $\sum_{k=1}^{n} \left(A_{ik}B_{kj}\right)$.

Definition 2. Given an $m \times n$ matrix A and a $n \times p$ matrix B, their product AB is defined to be the $m \times p$ matrix whose i, j entry is $\sum_{k=1}^{n} \left(A_{ik}B_{kj}\right)$.

So we multiply the entries of row j of A by those of column k of B, then add these together.

Example 3.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 & 0 & 0 \\ 2 & 1 & 0 & -1 \end{pmatrix} = \cdots$$

[Ask students about the other order BA.]

Proposition 4. *If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $[ST] = [S][T]$.*

Proof. This is true by the definition of matrix multiplication and the earlier calculation done as motivation. ☐

Let A be an $m \times n$ matrix.

- For $i = 1, \ldots, m$, let $A_{i,\cdot}$ denote row i of A, which is a $1 \times n$ matrix.
- For $j = 1, \ldots, n$, let $A_{\cdot,j}$ denote column j of A, which is an $m \times 1$ matrix.

The next few results give different interpretations of matrix multiplication. Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.

Lemma 5.

$$(AB)_{ij} = A_{i,\cdot} B_{\cdot,j}$$

for all $i = 1, \ldots, m$ and all $j = 1, \ldots, p$. [Draw picture of row and column.]

Proof. True by formula defining matrix multiplication. ☐

Lemma 6.

$$(AB)_{\cdot,j} = A(B_{\cdot,j})$$

for all $j = 1, \ldots, p$.

Proof. Exercise. Both are $m \times 1$ matrices. Check that their i^{th} entries are equal using the formula. [Draw picture applying A to each of the columns of B.] ☐

Lemma 7. *Suppose A is $m \times n$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is $n \times 1$. Then*

$$Ax = x_1 A_{\cdot 1} + \cdots + x_n A_{\cdot n} \,.$$

I.e., Ax is the linear combination of the columns of A with coefficients given by the entries of x.

Proof. Exercise. ☐

Lemma 8.

(a) For $j = 1, \ldots, p$, $(AB)_{\cdot,j}$ (column j) is a linear combination of the columns of A with coefficients from $B_{\cdot,j}$ (column j).

(b) For $i = 1, \ldots, m$, $(AB)_{i,\cdot}$ (row i) is a linear combination of the rows of B with coefficients from $A_{i,\cdot}$ (row i).

Proof. Exercise. [Draw picture of second part.] ☐

Definition 9.

- The *column space* of A, denoted $\mathrm{Col}(A)$, is the span of the columns of A. The *column rank* is the dimension of $\mathrm{Col}(A)$.
- The *row space* of A, denoted $\mathrm{Row}(A)$, is the span of the rows of A. The *row rank* is the dimension of $\mathrm{Row}(A)$.

We'll see that these two quantities are actually equal!

Definition 10. The *transpose* of a matrix A, denoted A^t, is obtained from A by interchanging rows and columns. I.e.,

$$(A^t)_{ij} = A_{ji} \,.$$

Lemma 11 (Column-row factorization)**.** *Suppose A is $m \times n$ and has column rank $c \in \mathbb{Z}_{\geq 1}$. Then there exist an $m \times c$ matrix C and a $c \times n$ matrix R such that $A = CR$. [Details left as an exercise.]*

Proof. The columns $A_{\cdot,1}, \ldots, A_{\cdot,n}$, each an $m \times 1$ matrix, span $\mathrm{Col}(A)$. By a previous result, this list can be reduced to a basis $v_1, \ldots, v_c$ of $\mathrm{Col}(A)$, which by definition must have length c. Use these as the columns of a $m \times c$ matrix C.

For $k = 1, \ldots, n$, column k of A is a linear combination of the columns of C (since these are a basis), so there exist scalars $R_{1k}, \ldots, R_{ck} \in \mathbb{F}$ such that

$$A_k = R_{1k}v_1 + \cdots + R_{ck}v_c\,.$$

Use the coefficients $R_{1k}, \ldots, R_{ck}$ as the entries of the k^{th} column of a $c \times n$ matrix R. Then $A = CR$. $\square$

Theorem 12 (Column rank = row rank)**.** *Suppose $A \in \mathrm{M}_{\mathrm{m\times n}}(\mathbb{F})$. Then the column rank and row rank of A are equal.*

Proof. Let c be the column rank of A. Let $A = CR$ be the column-row factorization of A given by the previous lemma, where C is $m \times c$ and R is $c \times n$. Since every row of A can be written as a linear combination of the rows of R, and R has c rows, then the row rank of A is $\leq c$, which is the column rank of A.

We obtain the reverse inequality by applying the same argument to A^t, which yields

$$\text{column rank of } A = \text{row rank of } A^t \leq \text{column rank of } A^t = \text{row rank of } A.$$

$\square$

Definition 13. The *rank* of a matrix is its column rank (= its row rank).

II.3. **Invertibility and Isomorphisms.**

Definition 14.

- $T \in \mathcal{L}(V, W)$ is *invertible* if there exists a linear map $S \in \mathcal{L}(W, V)$ such that $ST = I_V$ and $TS = I_W$.
- With the above notation, $S \in \mathcal{L}(W, V)$ is called an *inverse* of T.

Lemma 15. *An invertible linear map has a unique inverse.*

Proof idea. Given inverses S_1, S_2, then

$$\cdots = S_1TS_2 = \cdots$$

$\square$

If T is invertible, we denote its inverse by T^{-1}.

Example 16.

- Let

$$\begin{aligned} T : \mathbb{F}^2 &\to \mathbb{F}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} x + y \\ y \end{pmatrix}. \end{aligned}$$

Then T^{-1} is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x - y \\ y \end{pmatrix}.$$

[Write out at least one composition.]

- Let

$$\begin{aligned} R : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} \frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix}. \end{aligned}$$

(Rotation counterclockwise by $\pi/4$.) Claim: R^{-1} is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{pmatrix}.$$

Lemma 17. *A linear map $T \in \mathcal{L}(V, W)$ is invertible iff it is injective and surjective.*

Proof. ($\Rightarrow$): Assume T is invertible. One-to-one: Suppose $T(u) = T(v)$ for some $u, v \in V$. Applying T^{-1} to both sides, then

$$u = T^{-1}(T(u)) = T^{-1}(T(v)) = v\,.$$

Onto: Given $w \in W$, then $T^{-1}(w) \in V$ and $T(T^{-1}(w)) = w$, so $w \in \operatorname{img}(T)$.

($\Leftarrow$): Assume T is injective and surjective. Given $w \in W$, since T is surjective then there exists $v \in V$ such that $T(v) = w$. Suppose $v_1, v_2 \in V$ are both such preimages. Then

$$T(v_1) = w = T(v_2)$$

and since T is injective, then $v_1 = v_2$. Thus there is a *unique* $v \in V$ such that $T(v) = w$. Define the map $S : W \to V$ as follows: given $w \in W$, let $v \in V$ be the unique element such that $T(v) = w$. Defined $S(w) = v$. Then by definition we have $T(S(w)) = T(v) = w$, so $TS = I_W$. It remains to show $ST = I_V$.

Given $v \in V$, then

$$T((ST)(v)) = (TS)(T(v)) = I_W(T(v)) = T(v)\,.$$

Since T is one-to-one, then $(ST)(v) = v$. Thus $ST = I_V$.

It remains to show that S is linear. Suppose $w_1, w_2 \in W$. Then

$$T(S(w_1) + S(w_2)) = T(S(w_1)) + T(S(w_2)) = w_1 + w_2\,.$$

Now by definition, $S(w_1 + w_2)$ is the unique element that maps to $w_1 + w_2$ under T. Thus

$$S(w_1) + S(w_2) = S(w_1 + w_2)\,.$$

The proof that S respects scalar multiplication is similar. $\square$

Theorem 18. *Suppose V and W are finite-dimensional vector spaces with* $\dim(V) = \dim(W)$. *For any $T \in \mathcal{L}(V, W)$, the following are equivalent.*

(i) T is invertible.

(ii) T *is injective.*
(iii) T *is surjective.*

Proof. (ii) $\implies$ (iii): Suppose T is injective. Then $\dim(\ker(T)) = 0$. By the Rank-Nullity Theorem, then

$$\dim(V) = \cancelto{0}{\dim(\ker(T))} + \dim(\operatorname{img}(T)),$$

so $\dim(\operatorname{img}(T)) = \dim(V) = \dim(W)$. Then $\operatorname{img}(T) = W$, so W is surjective.

(iii) $\implies$ (ii): Similar.

By previous result, (i) $\iff$ (ii) and (iii), so they are all equivalent. $\square$

Remark 19. Warning! Finite-dimensionality is necessary in the above theorem. Consider the left-shift map

$$L : \mathbb{F}^\infty \to \mathbb{F}^\infty$$
$$(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots).$$

This map is surjective but not injective.

Proposition 20. *Suppose* V *and* W *are finite-dimensional vector spaces with* $\dim(V) = \dim(W)$*. Given* $T \in \mathcal{L}(V, W)$ *and* $S \in \mathcal{L}(W, V)$*, then* $ST = I \iff TS = I$.

Proof. ($\Rightarrow$): Assume $ST = I$. Given $v \in \ker(T)$, then $T(v) = 0$, so applying S, we have [start in middle]

$$v = I(v) = S(T(v)) = S(0) = 0.$$

Thus $\ker(T) = \{0\}$, so T is injective. Since V and W have the same dimension, then T is invertible by the previous result. Thus T^{-1} exists. Applying T^{-1} on the right to both sides of $I = ST$, we have [start in middle]

$$T^{-1} = IT^{-1} = STT^{-1} = S.$$

Thus $TS = TT^{-1} = I$, as desired.

($\Leftarrow$): Swap the roles of S and T. $\square$

II.3.1. *Isomorphic vector spaces.* The notion of isomorphism describes when two vector spaces are essentially "the same."

Definition 21. An *isomorphism* (of vector spaces) is an invertible linear map. Two vector spaces V and W are *isomorphic*, denoted $V \cong W$, if there is an isomorphism $V \to W$.

An isomorphism $T : V \to W$ is essentially just a relabeling: $v \in V$ is instead relabeled as $T(v) \in W$.

Q: How can we tell when two vector spaces are isomorphic?

Theorem 22 (Dimension determines isomorphism)**.** *Two finite-dimensional vector spaces over* $\mathbb{F}$ *are isomorphic iff they have the same dimension.*

Proof. Suppose V and W are finite-dimensional vector spaces.

($\Rightarrow$): Assume V and W are isomorphic. Then there exists an isomorphism $T : V \to W$. Then T is injective and surjective so

$$\ker(T) = \{0\} \qquad \text{and} \qquad \operatorname{img}(T) = W.$$

By Rank-Nullity, then [ask students]

$$\dim(V) = \cancelto{0}{\dim(ker(T))} + \dim(\operatorname{img}(T)) = \dim(W)\,.$$

($\Leftarrow$): Assume $\dim(V) = \dim(W)$. Let $v_1, \ldots, v_n$ be a basis for V and $w_1, \ldots, w_n$ be a basis for W. By a previous result, there is a unique linear map $T : V \to W$ such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$. Since $w_1, \ldots, w_n$ span W, then T is surjective. Either by Rank-Nullity, or by using the fact that $w_1, \ldots, w_n$ are linearly independent, T is injective. (Details left as exercise.) Thus T is injective and surjective, hence an isomorphism. □

Corollary 23. *Let V be an n-dimensional vector space. Then V is isomorphic to $\mathbb{F}^n$.*

Proof. Both have dimension n. □

Remark 24. We can also give an explicit isomorphism. Choose a basis $\mathcal{B} = (v_1, \ldots, v_n)$ for V and consider the coordinate vector map

$$\begin{aligned} \varphi_{\mathcal{B}} : V &\to \mathbb{F}^n \\ v &\mapsto [v]_{\mathcal{B}} \end{aligned}$$

and the linear map

$$\begin{aligned} S : \mathbb{F}^n &\to V \\ (a_1, \ldots, a_n) &\mapsto a_1 v_1 + \cdots + a_n v_n\,. \end{aligned}$$

Exercise: show these maps are mutually inverse isomorphisms.

Example 25. $\mathcal{P}_m(\mathbb{F})$ has dimension [ask students] $m+1$, hence is isomorphic to $\mathbb{F}^{m+1}$.

Proposition 26. *Suppose $\mathcal{B} := (v_1, \ldots, v_n)$ is a basis of V and $\mathcal{C} := (w_1, \ldots, w_m)$ is a basis of W (so $\dim(V) = n$ and $\dim(W) = m$). Then the map*

$$\begin{aligned} \mathcal{L}(V, W) &\to M_{m \times n}(\mathbb{F}) \\ T &\mapsto {}_{\mathcal{C}}[T]_{\mathcal{B}} \end{aligned}$$

is an isomorphism.

Proof. Exercise. □

Corollary 27. *Suppose V and W are finite-dimensional. Then $\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W)$.*

II.3.2. *Linear maps as matrices.*

Proposition 28 (Multiplication by a matrix is linear)**.** *Let $A \in \mathrm{M}_{\mathrm{m \times n}}(\mathbb{F})$. The left multiplication map*

$$\begin{aligned} L_A : \mathbb{F}^n &\to \mathbb{F}^m \\ v &\mapsto Av \end{aligned}$$

is linear.

Proof. Considering v as an $n \times 1$ matrix, this follows by properties of matrix multiplication. □

Let V and W be vector spaces with bases $\mathcal{B} := (v_1, \ldots, v_n)$ and $\mathcal{C} := (w_1, \ldots, w_m)$, respectively. Recall, for $T : V \to W$ linear, the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is given by

$$_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{pmatrix} | & & | \\ [T(v_1)]_{\mathcal{C}} & \cdots & [T(v_n)]_{\mathcal{C}} \\ | & & | \end{pmatrix}$$

Proposition 29. *With notation as above,*

$$[T(v)]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}[v]_{\mathcal{B}}$$

for all $v \in V$.

Proof. Given $v \in V$, there exist unique scalars $a_1, \ldots, a_n \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$. Since T is linear, then

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)\,.$$

Since the coordinate vector map is linear, then

$$\begin{aligned} [T(v)]_{\mathcal{C}} &= [a_1 T(v_1) + \cdots + a_n T(v_n)]_{\mathcal{C}} = a_1 [T(v_1)]_{\mathcal{C}} + \cdots + a_n [T(v_n)]_{\mathcal{C}} \\ &= \begin{pmatrix} | & & | \\ [T(v_1)]_{\mathcal{C}} & \cdots & [T(v_n)]_{\mathcal{C}} \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = {}_{\mathcal{C}}[T]_{\mathcal{B}}\, [v]_{\mathcal{B}}\,. \end{aligned}$$

□

The equality $[T(v)]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}\, [v]_{\mathcal{B}}$ can be stated by saying the following diagram "commutes."

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_{\mathcal{B}}"'] & W \arrow[d, "\varphi_{\mathcal{C}}"] \\ \mathbb{F}^n \arrow[r, "L_{_{\mathcal{C}}[T]_{\mathcal{B}}}"'] & \mathbb{F}^m \end{tikzcd}$$

[Draw image of v traveling both directions.]

Proposition 30. *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then the rank of T (i.e., $\dim(\operatorname{img}(T))$) is equal to the (column) rank of $[T]$.*

Proof. Exercise. □

II.3.3. *Change of basis.* Q: How does the natrix $_{\mathcal{C}}[T]_{\mathcal{B}}$ change if we change the bases $\mathcal{B}$ and $\mathcal{C}$?

Definition 31. Let $n \in \mathbb{Z}_{\geq 0}$. The *$n \times n$ identity matrix I* is the $n \times n$ matrix with 1s on the diagonal and 0s elsewhere:

$$I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Remark 32. We use I for both the identity operator and the identity matrix. With respect to *any* basis, the matrix of the identity operator I_V is I.

Definition 33. An $n \times n$ matrix A is *invertible* if there is a $n \times n$ matrix B such that $AB = BA = I$. We call B the *inverse* of A and denote it A^{-1}.

Lemma 34. *The inverse of a matrix is unique.*

Proof. Same as for linear maps. □

Theorem 35. *Let U, V, and W be vector spaces with bases $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$, respectively. Given $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then*

$$_{\mathcal{D}}[ST]_{\mathcal{B}} = {}_{\mathcal{D}}[S]_{\mathcal{C}}\, {}_{\mathcal{C}}[T]_{\mathcal{B}}\,.$$

Proof. Follows by the definition of matrix multiplication. □

Corollary 36 (Change of basis matrix)**.** *Suppose $\mathcal{B}$ and $\mathcal{C}$ are both bases for V. Then*

$$_{\mathcal{B}}[I]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{B}}^{-1}\,.$$

Proof.

$$I = {}_{\mathcal{B}}[I]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{C}}\, {}_{\mathcal{C}}[I]_{\mathcal{B}}\,.$$

□

Proposition 37 (Change of basis formula)**.** *Suppose $\mathcal{B}$ and $\mathcal{C}$ are both bases of V. Given $T \in \mathcal{L}(V)$, let $A := [T]_{\mathcal{B}}$, $B := [T]_{\mathcal{C}}$, and $C = {}_{\mathcal{B}}[I]_{\mathcal{C}}$. Then*

$$A = CBC^{-1}\,.$$

Proof.

$$_{\mathcal{B}}[T]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{C}}\, {}_{\mathcal{C}}[T]_{\mathcal{C}}\, {}_{\mathcal{C}}[I]_{\mathcal{B}} = ({}_{\mathcal{C}}[I]_{\mathcal{B}})^{-1}\, {}_{\mathcal{C}}[T]_{\mathcal{C}}\, {}_{\mathcal{C}}[I]_{\mathcal{B}}\,.$$

□

Definition 38. Two $n \times n$ matrices A and B are *similar* or *conjugate* if there is an invertible matrix P such that $B = PAP^{-1}$.