18.700 - LINEAR ALGEBRA, DAY 7
LINEAR MAPS, NULL SPACE, RANGE

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CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will learn the definition of dimension.
(2) Students will learn the definition of a linear map.
(3) Students will learn the definition of the null space and range of a linear map.
(4) Students will learn the Rank-Nullity Theorem.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.

(1) Laptop + adapter (2) Worksheets

(0:00)

II. Lesson Plan

Announcements: • Final exam: Monday, December 16, 1:30 - 4:30pm, 6-120

II.1. Last time.

- LI $\leq$ span
- Defined basis
- Can reduce spanning list to a basis
- Can extend a linearly independent list to a basis

II.2. 2C Dimension.

Theorem 1. *Any two bases of a finite-dimensional vector space have the same length.*

Proof. Suppose V is finite-dimensional and B_1 and B_2 are bases of V. Since B_1 is linearly independent and B_2 spans V, then by the LI $\leq$ span theorem, $\text{length}(B_1) \leq \text{length}(B_2)$. Reversing the roles of B_1 and B_2 yields the opposite inequality, so $\text{length}(B_1) = \text{length}(B_2)$. □

Definition 2. The *dimension* of a finite-dimensional vector space V is the length of any basis of V. Denoted $\dim(V)$.

Remark 3. This definition makes sense because of the previous theorem.

Lemma 4. *If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$.*

Proof. Exercise. (Similar to previous proof.) [Choose a basis B for U and C for V. Since B is LI and C spans V, can apply LI $\leq$ span result.] □

Proposition 5. *Suppose that V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim(V)$ is a basis.*

Proof. Let $n := \dim(V)$ and suppose $L := (v_1, \dots, v_n)$ are linearly independent. By the Extension Theorem, then L can be extended to a basis of V. But by the previous result, every basis of V has length n, so this must be the trivial extension, where no vectors are adjoined. Thus L was a basis of V to begin with. □

Example 6. Consider the list $(4, 2), (-1, 7)$ of vectors in $\mathbb{F}^2$. [Ask students why linearly independent.] Since $\dim(\mathbb{F}^2) = 2$ (consider the standard basis), then this list is a basis.

Corollary 7. *Suppose that V is finite-dimensional and U is a subspace of V such that $\dim(U) = \dim(V)$. Then $U = V$.*

Proof. Exercise. [Let $n = \dim(U) = \dim(V)$. Let B be a basis of U. Then B is linearly independent of size n, so is a basis of V by the Proposition. Then $U = \text{span}(B) = V$.] □

Proposition 8. *Suppose V is finite-dimensional. Then every spanning list S of V of length $\dim(V)$ is a basis of V.*

Proof. By a previous result, S can be reduced to a basis. However, every basis has length $\dim(V)$, so this reduction must be the trivial one, i.e., no vectors are removed from S. Thus S was a basis to begin with. □

Given subspaces V_1, V_2 with $V_1 \cap V_2 = \{0\}$, one can show that $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$. What if the sum is not direct?

Proposition 9. *Let V be finite-dimensional and V_1, V_2 be subspaces. Then*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)\,.$$

Remark 10. For a finite set S, let $\#S$ denote its cardinality, i.e., the number of elements in S. If S_1 and S_2 are finite sets, then

$$\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2)\,.$$

[Draw Venn diagram.]

Proof. Let $B := (v_1, \dots, v_m)$ be a basis for $V_1 \cap V_2$, so $\dim(V_1 \cap V_2) = m$. Since B is linearly independent, it can be extended to a basis

$$B_1 := (v_1, \dots, v_m, u_1, \dots, u_\ell)$$

of V_1, so $\dim(V_1) = m + \ell$. Similarly, B can be extended to a basis

$$B_2 := (v_1, \dots, v_m, w_1, \dots, w_n)$$

of V_2, so $\dim(V_2) = m + n$. We claim that

$$C := (v_1, \dots, v_m, u_1, \dots, u_\ell, w_1, \dots, w_n)$$

is a basis for $V_1 + V_2$. Note that if this holds, then

$$\begin{aligned} \dim(V_1 + V_2) &= m + \ell + n = (m + \ell) + (m + n) - m \\ &= \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)\,, \end{aligned}$$

which is what we want to show.

Observe that C is contained in $V_1 + V_2$ [ask students how to see $u_1 \in V_1 + V_2$]. Moreover, $\operatorname{span}(C)$ contains both V_1 and V_2, hence contains $V_1 + V_2$. Thus it remains to show that C is linearly independent. Suppose

$$a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell + c_1 w_1 + \cdots + c_n w_n = 0 \tag{$*$}$$

for some scalars $a_i, b_j, c_k \in \mathbb{F}$. Subtracting, then

$$c_1 w_1 + \cdots + c_n w_n = -(a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell) \in V_1\,.$$

By definition $w_1, \dots, w_n \in V_2$, so $c_1 w_1 + \cdots c_n w_n \in V_1 \cap V_2$. Since B is a basis of $V_1 \cap V_2$, then

$$c_1 w_1 + \cdots c_n w_n = d_1 v_1 + \cdots + d_m v_m$$

for some $d_1, \dots, d_m \in \mathbb{F}$. Subtracting, then

$$c_1 w_1 + \cdots c_n w_n - d_1 v_1 - \cdots - d_m v_m = 0\,.$$

But B_2 is basis, hence linearly independent, hence $c_1 = \cdots = c_n = 0 = d_1 = \cdots = d_m$. Then $(*)$ becomes

$$a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell = 0\,.$$

But B_1 is basis, hence linearly dependent, so $a_1 = \cdots = a_m = 0 = b_1 = \cdots = b_\ell$. Thus C is linearly independent, hence is a basis. $\square$

Here are some analogies between finite sets and finite dimensional vector spaces.

sets	vector spaces
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2)$ $= \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2)$ $= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2$ $\iff S_1 \cap S_2 = \emptyset$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2$ $\iff V_1 \cap V_2 = \{0\}$
$S_1 \cup \cdots \cup S_m$ is a disjoint union $\iff$ $\#(S_1 \cup \cdots \cup S_m) = \#S_1 + \cdots + \#S_m$	$V_1 + \cdots + V_m$ is a direct sum $\iff$ $\dim(V_1 + \cdots + V_m)$ $= \dim V_1 + \cdots + \dim V_m$

II.3. **Linear maps.** Linear maps are functions that preserve the vector space operations of addition and scalar multiplication. For this section, assume as usual that $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and let U, V, W be $\mathbb{F}$-vector spaces.

Definition 11. A function $T : V \to W$ is a *linear map* (or just *linear*) if

- $T(u + v) = T(u) + T(v)$ for all $u, v \in V$; and
- $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{F}$.

Remark 12. Also sometimes called *linear transformations*.

- The set of all linear maps $V \to W$ is denoted $\mathcal{L}(V, W)$.
- Let $\mathcal{L}(V) := \mathcal{L}(V, V)$.

Example 13.

- The zero linear map

$$\begin{aligned} 0 : V &\to W \\ v &\mapsto 0\,. \end{aligned}$$

- The identity map

$$\begin{aligned} I = I_V : V &\to V \\ v &\mapsto v\,. \end{aligned}$$

- Differentiation. Define

$$\begin{aligned} D : \mathcal{P}(\mathbb{R}) &\to \mathcal{P}(\mathbb{R}) \\ f &\mapsto f'\,. \end{aligned}$$

Since $(f + g)' = f' + g'$ and $(cf)' = cf'$, for all $f, g \in \mathcal{P}(\mathbb{R})$ and all $c \in \mathbb{R}$, then D is linear.

- Define
$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (2x - y + 3z, 7x + 5y - 6z) . \end{aligned}$$
Exercise to show that T is linear.

Lemma 14. *Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a* unique *linear map $T : V \to W$ such that*
$$T(v_i) = w_i$$
for each $i = 1, \ldots, n$.

Proof. Given $v \in V$, there exist unique scalars $c_1, \ldots, c_n \in \mathbb{F}$ such that $v = c_1 v_1 + \cdots + c_n v_n$. Define $T : V \to W$ by
$$T(v) = T(c_1 v_1 + \cdots + c_n v_n) := c_1 w_1 + \cdots + c_n w_n, .$$
This is well-defined because the scalars c_i are unique. Note that by taking $c_i = 1$ and $c_j = 0$ for $j \neq i$, we get $T(v_i) = w_i$, as required.

We now show that T is linear. Given $u, v \in V$, then there exist scalars $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{F}$ such that
$$\begin{aligned} u &= a_1 v_1 + \cdots + a_n v_n \\ v &= b_1 v_1 + \cdots + b_n v_n . \end{aligned}$$
Then
$$\begin{aligned} T(u + v) &= T(a_1 v_1 + \cdots + a_n v_n + b_1 v_1 + \cdots + b_n v_n) = T((a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n) \\ &= (a_1 + b_1) w_1 + \cdots + (a_n + b_n) w_n = a_1 w_1 + \cdots + a_n w_n + b_1 w_1 + \cdots + b_n w_n \\ &= T(a_1 v_1 + \cdots + a_n v_n) + T(b_1 v_1 + \cdots + b_n v_n) = T(u) + T(v) . \end{aligned}$$
Similarly, one can show that $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$. Thus T is linear. □

Remark 15. So a linear map is uniquely determined by its action on a basis.

Definition 16. Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The *sum* $S + T$ and scalar product λT are defined pointwise:
$$(S + T)(v) = S(v) + T(v) \qquad \text{and} \qquad (\lambda T)(v) = v$$
for all $v \in V$.

Lemma 17. *With notation as above, $S + T$ and λT are linear.*

Proposition 18. *With the operations of addition and scalar multiplication above, $\mathcal{L}(V, W)$ is a vector space.*

Proof. Exercise. [Ask students: what is the additive identity?] □

Definition 19. Given $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined as their composition:
$$(ST)(u) := (S \circ T)(u) = S(T(u))$$
for all $u \in U$.

Lemma 20. *With notation as above, ST is linear.*

Proposition 21 (Algebraic properties of linear maps)**.**

- $(T_1T_2)T_3 = T_1(T_2T_3)$ *whenever T_1, T_2, T_3 are linear maps such that the compositions are defined.*
- *Given $T \in \mathcal{L}(V, W)$, then $I_W T = T I_V$. [Ask students which identity operator.]*
- *For all $S, S_1, S_2 \in \mathcal{L}(V, W)$ and $T, T_1, T_2 \in \mathcal{L}(U, V)$, we have*

$$(S_1 + S_2)T = S_1T + S_2T \qquad \textit{and} \qquad S(T_1 + T_2) = ST_1 + ST_2 \,.$$

Remark 22. Composition of linear maps is not in general commutative!

Example 23. Let $V = \mathbb{F}^\infty$, the set of infinite sequences, and define

$$L : V \to V$$
$$(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots)$$

$$R : V \to V$$
$$(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots) \,.$$

Then

$$(LR)(x_1, x_2, x_3, \ldots) = L(R(x_1, x_2, x_3, \ldots)) = L(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots) \,,$$

but

$$(RL)(x_1, x_2, x_3, \ldots) = R(L(x_1, x_2, x_3, \ldots)) = R(x_2, x_3, \ldots) = (0, x_2, x_3, \ldots) \,,$$

Lemma 24. *Suppose $T : V \to W$ is a linear map. Then $T(0) = 0$.*

Proof. Exercise. □

II.4. **Null Spaces and Ranges aka Kernels and Images.**

II.4.1. *Null Spaces.*

Definition 25. Given $T \in \mathcal{L}(V, W)$, then *null space* or *kernel* of T, denoted (T) or $\ker(T)$ is

$$\ker(T) := \{v \in V : T(v) = 0\} \,.$$

The dimension of (T) is called the *nullity* of T.

[Draw picture of two blobs with kernel mapping to $0 \in W$.]

Lemma 26. *With notation as above, $\ker(T)$ is a subspace of V.*

Proof. Exercise. Apply subspace criterion. □

We can use the kernel to characterize when a linear map is one-to-one.

Definition 27. Let X and Y be sets. A function $f : X \to Y$ is *one-to-one* or *injective* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$.

Remark 28. The equivalent contrapositive statement: if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. So distinct inputs get mapped to distinct outputs under f. [Draw picture with blobs of one-to-one and not one-to-one functions.]

Proposition 29. *Let $T : V \to W$ be linear. Then T is injective iff $\ker(T) = \{0\}$.*

Proof. (⇒): Assume T is injective. Given $v \in \ker(T)$, then $T(v) = 0 = T(0)$. Since T is injective, then $v = 0$.

(⇐): Assume $\ker(T) = \{0\}$. Given $u, v \in V$ such that $T(u) = T(v)$, then

$$0 = T(u) - T(v) = T(u - v) = T(u) - T(v)$$

so $u - v \in \ker(T) = \{0\}$. Then $u - v = 0$, i.e., $u = v$. □

II.4.2. *Ranges.*

Definition 30. Let X and Y be sets and $f : X \to Y$ be a function. The *range* or *image* of f is

$$\operatorname{range}(f) = \operatorname{img}(f) = f(X) := \{f(x) : x \in X\} .$$

[Draw picture of blobs, showing that $\operatorname{img}(f)$ need not fill up all of Y.]

Definition 31. If $\operatorname{img}(f) = Y$, then f is *onto* or *surjective*.

Remark 32. Warning: You must specify the codomain for the notion of surjectivity to make sense! E.g., $f(x) = x^2$ as a function $\mathbb{R} \to \mathbb{R}$ or $\mathbb{R} \to [0, \infty)$.

Definition 33. Let $T : V \to W$ be linear. The dimension of $\operatorname{range}(T)$ is called the *rank* of T.

Lemma 34. *If $T : V \to W$ is linear, then* $\operatorname{img}(T)$ *is a subspace of W.*

Proof. Exercise. Apply subspace criterion. □

II.4.3. *Rank-nullity theorem.* The sizes of the kernel and the image are inversely correlated. E.g., the zero map $0 : V \to W$ has large null space [ask students]—all of V—and small range—just $\{0\}$. On the other hand, the identity map $I : V \to V$ has small kernel [ask students]—just $\{0\}$—and large image—all of V. This relationship is captured precisely in the following result.

Theorem 35 (Rank-Nullity Theorem)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then* $\operatorname{img}(T)$ *is also finite-dimensional and*

$$\dim(V) = \dim(\ker(T)) + \dim(\operatorname{img}(T)) .$$

In words, the dimension of the domain of T is equal to the sum of the nullity and rank of T.

Proof. Let $u_1, \dots, u_m$ be a basis of $\ker(T)$. By the Extension Theorem, we can extend this to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. Thus $\dim(\ker(T)) = m$ and $\dim(V) = m + n$, so it suffices to show that $\dim(\operatorname{img}(T)) = n$.

We claim that $T(v_1), \dots, T(v_n)$ is a basis for $\operatorname{img}(T)$. [Ask students why we don't include any u_i.] Given $v \in V$, then

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

for some scalars $a_i, b_j \in \mathbb{F}$. Applying T, we have

$$T(v) = T\left(\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j\right) = \sum_i a_i \cancelto{0}{T(u_i)} + \sum_j b_j T(v_j)$$

since the u_i all map to 0 since they are in $\ker(T)$. Thus $T(v_1), \dots, T(v_n)$ spans $\operatorname{img}(T)$, hence it is finite-dimensional.

It remains to show they are linearly independent. Suppose there exists $c_1, \ldots, c_n \in \mathbb{F}$ such that

$$0 = c_1 T(v_1) + \cdots + c_n T(v_n) = T(c_1 v_1 + \cdots + c_n v_n) .$$

Then $\sum_{k=1}^{n} c_k v_k \in \ker(T)$, so there exist $d_1, \ldots, d_m \in \mathbb{F}$ such that

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots d_m u_m .$$

Then

$$0 = d_1 u_1 + \cdots d_m u_m - c_1 v_1 - \cdots - c_n v_n .$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is a basis of V, hence linearly independent, then $0 = c_1 = \cdots, c_n = d_1 = \cdots = d_m$. Thus $T(v_1), \ldots, T(v_n)$ is linearly independent, hence is a basis of $\operatorname{img}(T)$. □

Corollary 36. *Suppose V and W are finite-dimensional vector spaces with* $\dim(V) > \dim(W)$. *Then no linear map $T : V \to W$ is injective.*

Proof. By the Rank-Nullity Theorem, then

$$\dim(\ker(T)) = \dim(V) - \dim(\operatorname{img}(T)) .$$

Since $\operatorname{img}(T) \subseteq W$, then $\dim(\operatorname{img}(T)) \leq \dim(W)$, so $-\dim(W) \leq -\dim(\operatorname{img}(T))$. Then

$$\dim(\ker(T)) = \dim(V) - \dim(\operatorname{img}(T)) \geq \dim(V) - \dim(W) > 0 .$$

Thus $\ker(T) \neq \{0\}$, so T is not injective. □

Corollary 37. *Suppose V and W are finite-dimensional vector spaces with* $\dim(V) < \dim(W)$. *Then no linear map $T : V \to W$ is surjective.*

Proof. Exercise. Similar to the above. □