18.700 - LINEAR ALGEBRA, DAY 6
BASES AND DIMENSION

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CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will learn the definitions of basis and dimension.
(2) Students will learn the uniqueness of representing a vector with respect to a basis.
(3) Students will learn criteria for checking if a list of vectors is a basis.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.

(1) Laptop + adapter (2) Worksheets

(0:00)

II. Lesson Plan

II.1. Last time.

- Gave a criterion for checking with $V_1 + V_2$ is a direct sum.
- Gave definition of spanning list and linearly independent list of vectors.
- Defined polynomials.
- Proved the Linear Dependence Lemma: if $v_1, \dots, v_m$ is linearly dependent, then we can express one of the vectors as a linear combination of the previous ones.

II.2. Worksheet.

II.3. Results on span and linear independence.

Theorem. *(LI $\leq$ span) Let V be a finite dimensional vector space. Then the length of every linearly independent list of vectors in V is $\leq$ the length of every spanning list of vectors.*

Proof. Suppose $L := (u_1, \dots, u_m)$ is a linearly independent list in V and $S := (w_1, \dots, w_n)$ is a list spannning V. Goal: $m \leq n$. We show this by recursively replacing the w_i by the u_j, one by one.

Base case: Since S spans V, then

$$u_1, w_1, \dots, w_n$$

must be linearly dependent. By the Linear Dependence Lemma, then one of the above vectors in the list can be written as a linear combination of the previous vectors in the list. Since L is linearly independent, then $u_1 \neq 0$, so it's not u_1. Thus it must be the case that there exists r such that

$$\operatorname{span}(u_1, w_1, \dots, \widehat{w_r}, \dots, w_n) = \operatorname{span}(u_1, w_1, \dots, w_r, \dots, w_n) = V\,.$$

Remove w_r from S, so now S is

$$u_1, w_1, \dots, \widehat{w_r}, \dots, w_n\,,$$

and note that it still has length n and still spans V.

Recursive step: Let $k \in \{2, \dots, m\}$ and suppose we have already replaced $k-1$ of the w_i with $u_1, \dots, u_{k-1}$, producing a list S of length n that still spans V. Then the list S looks like $u_1, \dots, u_{k-1}$ followed by any remaining w_i (possibly none). Adjoin u_k to S just after u_{k-1}; since the list already spanned V, so now it is linearly dependent. By the Linear Dependence Lemma, then one of the vectors in S can be written as a linear combination of the vectors preceding it. Again, since $u_1, \dots, u_k$ is linearly independent, this vector cannot be one of the u_j. Thus there must be at least one remaining w_i still in the list. So one of the remaining w_i, say w_r, can be written as a linear combination of the previous vectors. We remove w_r, which doesn't change the span, and redefine S to be this modified list.

After step m, we have adjoined all the u_j to S, a list of length n. Note that S still spans V, and now looks like $u_1, \dots, u_m$ and then any remaining w_i (possibly none). At each step we adjoined a u_j, and the Linear Dependence Lemma implied the existence of a w_i to remove. Thus $m \leq n$. □

Proposition 1. *Every subspace of a finite-dimensional vector space is finite-dimensional.*

Proof. Let V be a finite-dimensional vector space and U a subspace of V. Goal: Find a (finite) spanning list for U. We do this recursively.

Base case: If $U = \{0\}$, then 0 generates U and we're done. Otherwise, $U \neq \{0\}$, so we can choose a nonzero vector $u_1 \in U$.

Recursive step: Suppose we have already chosen $k-1$ nonzero vectors $u_1, \dots, u_{k-1}$. If $U = \operatorname{span}(u_1, \dots, u_{k-1})$, we're done. Otherwise, $U \neq \operatorname{span}(u_1, \dots, u_{k-1})$, so we can choose $u_k \notin \operatorname{span}(u_1, \dots, u_{k-1})$. As you'll show on the next pset, this implies that $u_1, \dots, u_{k-1}, u_k$ is linearly independent.

Thus at each step k, we produce a linearly independent list of length k in U. But by the previous theorem, no linearly independent list can be longer than a spanning list of V. Thus the process eventually terminates, at which point we have a finite spanning list for U. □

II.4. **2B Bases.** We now combine the properties of linearly independent and spanning lists.

Definition 1. A list $v_1, \dots, v_n \in V$ is a *basis* of V iff every $v \in V$ can be written *uniquely* as a linear combination

$$v = a_1 v_1 + \cdots + a_n v_n$$

with $a_1, \dots, a_n \in \mathbb{F}$.

Theorem. *A list $v_1, \dots, v_n \in V$ is a basis of V iff it is linearly independent and spans V.*

Proof. Let $B = (v_1, \dots, v_n)$. ($\Rightarrow$): Assume B is a basis. Since every $v \in V$ can be written as a linear combination of $v_1, \dots, v_n$, then B spans V. [Ask students how to check linear independence.] Suppose

$$a_1 v_1 + \cdots + a_n v_n = 0$$

for some $a_1, \dots, a_n \in \mathbb{F}$. Since

$$0\, v_1 + \cdots + 0\, v_n = 0$$

then by uniqueness, we must have $a_1 = \cdots = a_n = 0$.

($\Leftarrow$): Now assume B is linearly independent and spans V. Given $v \in V$, since B spans V, then

$$v = a_1 v_1 + \cdots + a_n v_n$$

for some $a_1, \dots, a_n \in \mathbb{F}$. We aim to show this expression is unique. Given $b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = b_1 v_1 + \cdots + b_n v_n\,,$$

then by subtracting, we have

$$0 = v - v = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n\,.$$

Since B is linearly independent, then $a_1 - b_1 = \cdots = a_n - b_n = 0$, i.e., $a_1 = b_1, \dots, a_n = b_n$. □

Theorem. *Every spanning list S of a finite-dimensional vector space V contains a basis of V.*

Proof. Suppose $S := (v_1, \dots, v_n)$ spans V. We give an algorithm to remove vectors from S until we obtain a basis. Start with $B := S$.

If $v_1 = 0$, then delete v_1 from B; otherwise, leave B unchanged.

For $k \in \{2, \dots, n\}$, if $v_k \in \text{span}(v_1, \dots, v_{k-1})$, then delete v_k from B. Otherwise, leave B unchanged.

After step n, we claim that B is a basis of V. Since the original list spanned V and we only removed vectors that were contained in the span of the previous vectors, then B spans V. Moreover, by construction of the algorithm, no vector in B is in the span of the previous ones. Thus B is linearly independent by the Linear Dependence Lemma. Thus B is a basis. □

Corollary 1. *Every finite-dimensional vector space V has a basis.*

Proof. By definition, V has a finite spanning list S. By the previous result, we can find a basis contained in S. □

Theorem (Extension Theorem)**.** *Every linearly independent list L of vectors in a finite dimensional vector space V can be extended to a basis of V.*

Proof. Write $L := (u_1, \dots, u_m)$. Let $S := (w_1, \dots, w_n)$ be a list of vectors spanning V. Then their concatenation

$$u_1, \dots, u_m, w_1, \dots, w_n$$

spans V. Apply the algorithm from the proof of the previous theorem to obtain a basis B of V. Since L is linearly independent, then none of $u_1, \dots, u_m$ get deleted during the algorithm. Thus B contains $u_1, \dots, u_m$ and thus extends L. □

Proposition 2. *Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.*

Proof. Since V is finite-dimensional, then so is U by a previous result. Let $u_1, \dots, u_m$ be a basis of U. By the previous result, then it can be extended to a basis

$$B = (u_1, \dots, u_m, w_1, \dots, w_n)$$

of V. Let $W = \text{span}(w_1, \dots, w_n)$. We claim that $V = U \oplus W$. By our direct sum criterion, it suffices to show that $U + W = V$ and $U \cap W = \{0\}$.

Given $v \in V$, since B is a basis of V, then there exist $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{u} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{w} .$$

Since U and W are subspaces, then they are closed under linear combinations, so $u \in U$ and $w \in W$, and hence $v = u + w \in U + W$.

Now suppose $v \in U \cap W$. Since $v \in U$, then

$$v = a_1 u_1 + \cdots + a_m u_m$$

for some $a_1, \dots, a_m \in \mathbb{F}$, and since $v \in W$, then

$$v = b_1 w_1 + \cdots + b_n w_n$$

for some $b_1, \dots, b_n \in \mathbb{F}$. Subtracting, we have

$$0 = v - v = a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n .$$

But B is a basis, hence linearly independent, so $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Thus $v = 0$. □

II.5. **2C Dimension.**

Theorem. *Any two bases of a finite-dimensional vector space have the same length.*

Proof. Suppose V is finite-dimensional and B_1 and B_2 are bases of V. Since B_1 is linearly independent and B_2 spans V, then by the LI $\leq$ span theorem, $\text{length}(B_1) \leq \text{length}(B_2)$. Reversing the roles of B_1 and B_2 yields the opposite inequality, so $\text{length}(B_1) = \text{length}(B_2)$. □

Definition 2. The *dimension* of a finite-dimensional vector space V is the length of any basis of V. Denoted $\dim(V)$.

Lemma 1. *If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$.*

Proof. Exercise. (Similar to previous proof.) □

Proposition 3. *Suppose that V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim(V)$ is a basis.*

Proof. Let $n := \dim(V)$ and suppose $L := (v_1, \ldots, v_n)$ are linearly independent. By the Extension Theorem, then L can be extended to a basis of V. But by the previous result, every basis of V has length n, so this must be the trivial extension, where no vectors are adjoined. Thus L was a basis of V to begin with. □

Example 1. Consider the list $(4, 2), (-1, 7)$ of vectors in $\mathbb{F}^2$. [Ask students why linearly independent.] Since $\dim(\mathbb{F}^2) = 2$ (consider the standard basis), then this list is a basis.

Corollary 2. *Suppose that V is finite-dimensional and U is a subspace of V such that $\dim(U) = \dim(V)$. Then $U = V$.*

Proof. Exercise. □

Proposition 4. *Suppose V is finite-dimensional. Then every spanning list S of V of length $\dim(V)$ is a basis of V.*

Proof. By a previous result, S can be reduced to a basis. However, every basis has length $\dim(V)$, so this reduction must be the trivial one, i.e., no vectors are removed from S. Thus S was a basis to begin with. □

Given subspaces V_1, V_2, one can show that $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$. What if the sum is not direct?

Proposition 5. *Let V be finite-dimensional and V_1, V_2 be subspaces. Then*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)\,.$$

Proof. Let $B := (v_1, \ldots, v_m)$ be a basis for $V_1 \cap V_2$, so $\dim(V_1 \cap V_2) = m$. Since B is linearly independent, it can be extended to a basis

$$B_1 := (v_1, \ldots, v_m, u_1, \ldots, u_\ell)$$

of V_1, so $\dim(V_1) = m + \ell$. Similarly, it can be extended to a basis

$$B_2 := (v_1, \ldots, v_m, w_1, \ldots, w_n)$$

of V_2, so $\dim(V_2) = m + n$. We claim that

$$C := (v_1, \ldots, v_m, u_1, \ldots, u_\ell, w_1, \ldots, w_n)$$

is a basis for $V_1 + V_2$. Note that if this holds, then

$$\dim(V_1 + V_2) = m + \ell + n = (m + \ell) + (m + n) - m = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)\,,$$

which is what we want to show.

Observe that C is contained in $V_1 + V_2$ [ask students how to see $u_1 \in V_1 + V_2$]. Moreover, $SSpan(C)$ contains both V_1 and V_2, hence contains $V_1 + V_2$. Thus it remains to show that C is linearly independent. Suppose

$$a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell + c_1 w_1 + \cdots + c_n w_n = 0 \qquad (*)$$

for some scalars $a_i, b_j, c_k \in \mathbb{F}$. Subtracting, then

$$c_1 w_1 + \cdots + c_n w_n = -(a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell) \in V_1\,.$$

By definition $w_1, \ldots, w_n \in V_2$, so $c_1 w_1 + \cdots c_n w_n \in V_1 \cap V_2$. Since B is a basis of $V_1 \cap V_2$, then

$$c_1 w_1 + \cdots c_n w_n = d_1 v_1 + \cdots + d_m v_m$$

for some $d_1, \ldots, d_m \in \mathbb{F}$. Subtracting, then

$$c_1 w_1 + \cdots c_n w_n - d_1 v_1 - \cdots - d_m v_m = 0\,.$$

But B_2 is basis, hence linearly independent, hence $c_1 = \cdots = c_n = 0 = d_1 = \cdots = d_m$. Then $(*)$ becomes

$$a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_\ell u_\ell = 0\,.$$

But B_1 is basis, hence linearly dependent, so $a_1 = \cdots = a_m = 0 = b_1 = \cdots = b_\ell$. Thus C is linearly independent, hence is a basis. □

Remark 1. For a finite set S, let $\#S$ denote its cardinality, i.e., the number of elements in S. If S_1 and S_2 are finite sets, then

$$\#(S_1 \cup S_2) = \#S_1 + \#S_2 - \#(S_1 \cap S_2)\,.$$

[Draw Venn diagram.]

Here are some analogies between finite sets and finite dimensional vector spaces.

sets	**vector spaces**
S is a finite set	V is a finite-dimensional vector space
$\#S$	$\dim V$
for subsets S_1, S_2 of S, the union $S_1 \cup S_2$ is the smallest subset of S containing S_1 and S_2	for subspaces V_1, V_2 of V, the sum $V_1 + V_2$ is the smallest subspace of V containing V_1 and V_2
$\#(S_1 \cup S_2)$ $= \#S_1 + \#S_2 - \#(S_1 \cap S_2)$	$\dim(V_1 + V_2)$ $= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$
$\#(S_1 \cup S_2) = \#S_1 + \#S_2$ $\iff S_1 \cap S_2 = \emptyset$	$\dim(V_1 + V_2) = \dim V_1 + \dim V_2$ $\iff V_1 \cap V_2 = \{0\}$
$S_1 \cup \cdots \cup S_m$ is a disjoint union $\iff$ $\#(S_1 \cup \cdots \cup S_m) = \#S_1 + \cdots + \#S_m$	$V_1 + \cdots + V_m$ is a direct sum $\iff$ $\dim(V_1 + \cdots + V_m)$ $= \dim V_1 + \cdots + \dim V_m$