18.700 - LINEAR ALGEBRA, DAY 5
SPAN AND LINEAR INDEPENDENCE

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CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will learn the definition of span and spanning list.
(2) Students will learn the definition of linear independence.
(3) Students will learn the defintion of a polynomial.
(4) Students will learn the Linear Independence Lemma and the Replacement Theorem.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.

(1) Laptop + adapter (2) Worksheets

II. Lesson Plan

(0:00)

Announcements: • Second pset due this evening. • Pset Partners

II.1. **Last time.**

- Defined subspaces and gave a criterion to check if a subset is a subspace.
- Applied the subspace criterion to examples.
- Defined sum and direct sum of vector spaces.
- Started to give criterion for checking if a sum is direct.

(0:05) II.2. **Direct sum wrap up.**

Definition 1. Let $V_1, \ldots, V_m$ be subspaces of V.

- Their *sum* is

$$V_1 + \cdots + V_m := \{v_1 + \cdots + v_m : v_1 \in V_1, \ldots, v_m \in V_m\}.$$

- The sum is a *direct*, denoted $V_1 \oplus \cdots \oplus V_m$, if each $v \in V_1 + \cdots + V_m$ [ask students] can be written *uniquely* as

$$v = v_1 + \cdots + v_m$$

with $v_i \in V_i$ for $i = 1, \ldots, m$.

Lemma 1. *With notation as above,* $V_1 + \cdots + V_m$ *is a subspace of* V.

Lemma 2. $V_1 + \cdots + V_m$ *is a direct sum iff the only way to write*

$$0 = v_1 + \cdots + v_m$$

with $v_i \in V_i$ *for all* i *is by taking* $v_i = 0$ *for all* i.

Proof. ($\Rightarrow$): Follows from the definition of direct sum, taking $v = 0$.
($\Leftarrow$): Given $v \in V_1 + \cdots + V_m$, suppose

$$v = v_1 + \cdots + v_m$$
$$v = u_1 + \cdots + u_m$$

where $u_i, v_i \in V_i$ for all $i = 1, \ldots, m$. Then

$$0 = v - v = (v_1 + \cdots + v_m) - (u_1 + \cdots + u_m) = \cdots = (v_1 - u_1) + \cdots + (v_m - u_m).$$

But we also have $0 = 0 + \cdots + 0$ and since this expression for 0 is unique, we must have $v_i - u_i = 0$, i.e., $v_i = u_i$ for all i. Thus the expression for v is unique. □

Proposition 1. $V_1 + V_2$ *is a direct sum iff* $V_1 \cap V_2 = \{0\}$.

Proof. Exercise. □

[Ask students what $U_1 \cap U_2$ was in first example, where

$$U_1 := \{(s, s, t, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}$$
$$U_2 := \{(s, s, s, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}.$$

]

II.3. **2A. Span and Linear Independence.** Throughout, assume that V is an $\mathbb{F}$-vector space.

Remark 1. We will usually write lists of vectors without parentheses, e.g., $v_1, v_2, \ldots, v_m$.

Definition 2. A *linear combination* of a list $v_1, \ldots, v_m \in V$ is an expression of the form

$$a_1 v_1 + \cdots + a_m v_m$$

where $a_1, \ldots, a_m \in \mathbb{F}$.

Example 1.

- Let $u_1 := (1, -1, 3)$ and $u_2 := (2, -1, 4)$. Then
  $$3u_1 - 2u_2 = 3(1, -1, 3) - 2(2, -1, 4) = (3, -3, 9) + (-4, 2, -8) = (-1, -1, 1)$$
  so $(-1, -1, 1)$ is a linear combination of u_1 and u_2.
- Is $w := (1, -2, -1)$ a linear combination of u_1 and u_2? [Ask students what this means in symbols.] I.e., do there exist $c_1, c_2 \in \mathbb{F}$ such that $c_1 u_1 + c_2 u_2 = w$?
  $$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -2 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -6 \end{array}\right)$$

Definition 3. The *span* of a list of vectors $v_1, \ldots, v_m \in V$, denoted $\operatorname{span}(v_1, \ldots, v_m)$, is the set of all their linear combinations. I.e.,

$$\operatorname{span}(v_1, \ldots, v_m) := \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}.$$

The span of the empty list $()$ is defined to be $\{0\}$.

Example 2. With notation as in the previous example,

$$(-1, -1, 1) \in \operatorname{span}(u_1, u_2) \qquad \text{and} \qquad (1, -2, -1) \notin \operatorname{span}(u_1, u_2).$$

Lemma 3. *With notation as above,* $\operatorname{span}(v_1, \ldots, v_m)$ *is the* smallest *subspace of V containing $v_1, \ldots, v_m$.*

Proof. Exercise. ☐

Definition 4. If $\operatorname{span}(v_1, \ldots, v_m) = V$, then we say that $v_1, \ldots, v_m$ *span* or *generate* V.

Definition 5. A vector space is *finite-dimensional* if it is spanned by some (finite) list of vectors. Otherwise, it is *infinite-dimensional*.

II.3.1. *Polynomials.*

Definition 6. A function $p : \mathbb{F} \to \mathbb{F}$ is a *polynomial (function)* with coefficients in $\mathbb{F}$ if there exist $a_0, \ldots, a_m \in \mathbb{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$. We denote the set of all polynomial with coefficients in $\mathbb{F}$ by $\mathcal{P}(\mathbb{F})$ or $\mathbb{F}[z]$.

Lemma 4. $\mathcal{P}(\mathbb{F})$ *is an* $\mathbb{F}$*-vector space.*

Proof. Exercise. ☐

Definition 7.

- A polynomial $p \in \mathcal{P}(\mathbb{F})$ has *degree d* if there exist scalars $a_0, \ldots, a_d \in \mathbb{F}$ with $a_d \neq 0$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
for all $z \in \mathbb{F}$.
- The degree of p is denoted $\deg(p)$.
- By convention, $\deg(0) = -\infty$.
- Let $\mathcal{P}_m(\mathbb{F})$ be the set of all polynomials of degree $\leq m$.

Lemma 5.
- *$\mathcal{P}_m(\mathbb{F})$ is finite-dimensional.*
- *$\mathcal{P}(\mathbb{F})$ is infinite-dimensional.*

Proof.
- [Ask students.] $\mathcal{P}_m(\mathbb{F}) = \text{span}(1, z, \ldots, z^m)$.
- Given any finite list $p_1, p_2, \ldots, p_k$ of polynomials, let M be the maximum of their degrees. Then every polynomial in $\text{span}(p_1, \ldots, p_k)$ has degree at most M. But then $z^{M+1} \notin \text{span}(p_1, \ldots, p_k)$, so the list does not span $\mathcal{P}(\mathbb{F})$. This shows that no finite list spans $\mathcal{P}(\mathbb{F})$, so it is infinite-dimensional.

□

Remark 2. $\mathbb{F}^\infty$ is also infinite-dimensional. Can you find a proof of this similar to that for $\mathcal{P}(\mathbb{F})$?

II.3.2. *Linear independence.* Q: Can we characterize *minimal* spanning sets?

Q: Given $v_1, \ldots, v_m$ and $v \in \text{span}(v_1, \ldots, v_m)$, when is there a *unique* list of scalars $a_1, \ldots, a_m \in \mathbb{F}$ such that
$$v = a_1 v_1 + \cdots + a_m v_m\,?$$
(Similar to the definition of direct sum.)

If there is another such list of scalars $b_1, \ldots, b_m$, then
$$a_1 v_1 + \cdots + a_m v_m = v = b_1 v_1 + \cdots + b_m v_m\,.$$
Subtracting, then
$$0 = (a_1 - b_1) v_1 + \cdots + (a_m - b_m) v_m\,.$$

Definition 8. A list $v_1, \ldots, v_m \in V$ is *linearly dependent* if there exist scalars $a_1, \ldots, a_m \in \mathbb{F}$, not all 0, such that
$$a_1 v_1 + \cdots a_m v_m = 0\,.$$
A list is *linearly independent* if is not linearly dependent, i.e., if the only choice of scalars $a_1, \ldots, a_m \in \mathbb{F}$ such that
$$a_1 v_1 + \cdots a_m v_m = 0$$
is $a_1 = \cdots = a_m = 0$.

By convention, the empty list () is linearly independent.

Remark 3.
- A list v of length one in a vector space is linearly dependent iff [ask students] $v = 0$.
- A list v, w of length two is linearly dependent iff v and w are scalar multiples of each other.

Lemma 6. *(Linear Dependence Lemma) Suppose $v_1, \dots, v_m \in V$ is a linearly dependent list. Then there exists $k \in \{1, 2, \dots, m\}$ such that*

$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1})\,.$$

Furthermore, removing v_k leaves the span unchanged, i.e,

$$\operatorname{span}(v_1, \dots, \widehat{v_k}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m)\,.$$

Proof. Since $v_1, \dots, v_m$ is linearly dependent, then [ask students] there exist $a_1, \dots a_m$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0\,. \tag{1}$$

Let k be the largest element of $\{1, \dots, m\}$ such that $a_k \neq 0$. Subtracting to the other side and dividing by a_k, we have

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1}\,. \tag{2}$$

The last statement is left as an exercise. The idea is, in any linear combination of $v_1, \dots, v_m$, one can replace v_k by the RHS of (2). □

II.4. **Worksheet.**

II.5. **Results on span and linear independence.**

Theorem. *(Replacement Theorem, LI $\leq$ span) Let V be a finite dimensional vector space. Then the length of every linearly independent list of vectors in V is $\leq$ the length of every spanning list of vectors.*

Proof. Suppose $B := (u_1, \dots, u_m)$ is a linearly independent list in V and $C := (w_1, \dots, w_n)$ is a list spannning V. Goal: $m \leq n$. We show this by recursively replacing the w_i by the u_j, one by one.

Base case: Since C spans V, then

$$u_1, w_1, \dots, w_n$$

must be linearly dependent. By the Linear Dependence Lemma, then one of the above vectors in the list can be written as a linear combination of the previous vectors in the list. Since B is linearly independent, then $u_1 \neq 0$, so it's not u_1. Thus it must be the case that there exists r such that

$$\operatorname{span}(u_1, w_1, \dots, \widehat{w_r}, \dots, w_n) = \operatorname{span}(u_1, w_1, \dots, w_r, \dots, w_n) = V\,.$$

Redefine C to be

$$u_1, w_1, \dots, \widehat{w_r}, \dots, w_n\,,$$

and note that it still has length n.

Recursive step: Let $k \in \{2, \dots, m\}$ and suppose we have already replaced $k-1$ of the w_i with $u_1, \dots, u_{k-1}$. Then the list C looks like $u_1, \dots, u_{k-1}$ followed by the remaining w_i. Adjoin u_k to C just after u_{k-1}. By the Linear Dependence Lemma, then one of the vectors in C can be written as a linear combination of the vectors preceding it. Again, since $u_1, \dots, u_k$ is linearly independent, this vector cannot be one of the u_i. Thus it must be one of the w_j, say w_r. We remove w_r, which doesn't change the span, and redefine C to be this modified list.

After step m, we have adjoined all the u_j to C, a list of length n. Note that C still spans V, and now looks like $u_1, \ldots, u_m$ and then the remaining w_i. At each step we adjoined a u_j, and the Linear Dependence Lemma implied the exitence of a w_i to remove. Thus $m \leq n$. □

Proposition 2. *Every subspace of a finite-dimensional vector space is finite-dimensional.*

Proof. Let V be a finite-dimensional vector space and U a subspace of V. Goal: Find a (finite) spanning list for U. We do this recursively.

Base case: If $U = \{0\}$, then 0 generates U and we're done. Otherwise, $U \neq \{0\}$, so we can choose a nonzero vector $u_1 \in U$.

Recursive step: Suppose we have already chosen $k-1$ nonzero vectors $u_1, \ldots, u_{k-1}$. If $U = \text{span}(u_1, \ldots, u_{k-1})$, we're done. Otherwise, $U \neq \text{span}(u_1, \ldots, u_{k-1})$, so we can choose $u_k \notin \text{span}(u_1, \ldots, u_{k-1})$. As you'll show on the next pset, this implies that $u_1, \ldots, u_{k-1}, u_k$ is linearly independent.

Thus at each step k, we produce a linearly independent list of length k in U. But by the Replacement Theorem, no linearly independent list can be longer than a spanning list of V. Thus the process eventually terminates, at which point we have a finite spanning list for U. □