18.700 - LINEAR ALGEBRA, DAY 4
SUBSPACES

SAM SCHIAVONE

CONTENTS

I. Pre-class Planning 1
I.1. Goals for lesson 1
I.2. Methods of assessment 1
I.3. Materials to bring 1
II. Lesson Plan 2
II.1. Last time 2
II.2. Worksheet 2
II.3. Subspaces 2

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will start or continue learning how to write proofs.
(2) Students will learn the subspace criterion.
(3) Students will apply the subspace criterion to examples.
(4) Students will learn the definitions of the sum of a finite collection of subspaces.
(5) Students will learn the definition of a direct sum of a finite collection of subspaces.
(6) Students will learn how to check if a sum of subspaces is direct.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.
(1) Laptop + adapter (2) Worksheets

(0:00)

II. Lesson Plan

Announcements: • Second pset due Wednesday. • Mention Velleman's *How To Prove It*.

II.1. **Last time.**

- Recalled the algebraic properties of $\mathbb{R}^n$ and $\mathbb{C}^n$.
- Abstracted these properties and gave the definition of a vector space.
- Proved some elementary properties of vector spaces.
- Considered some examples of vector spaces, such as $\mathbb{F}^n$, $\mathbb{F}^\infty$, and $\mathbb{F}^S$.

(0:05) II.2. **Worksheet.**

II.3. **Subspaces.** Given a vector space V, it is often useful to consider smaller vector spaces that are contained inside V. We call these subspaces. [Draw picture of a line
(0:15) through the origin inside $\mathbb{R}^2$.]

Definition 1. A subset $U \subseteq V$ is called a *(vector) subspace* if U is also a vector space when considered with the same addition, scalar multiplication, and additive identity as V.

Remark 1.

- Also sometimes known as *linear subspaces*.
- $\{0\}$ and V itself are always subspaces of a vector space V.

To check if a subset $U \subseteq V$ is a subspace, it's not necessary to check all the vector space axioms. We get some for free from knowing already that V is a vector space.

Proposition 1 (Subspace criterion)**.** *A subset $U \subseteq V$ is a subspace of V iff U satisfies the following.*

(i) $0 \in U$.
(ii) U is closed under addition, i.e., if $u, v \in U$, then $u + v \in U$.
(iii) U is closed under scalar multiplication, i.e., if $a \in \mathbb{F}$ and $u \in U$, then $au \in U$.

Proof. Suppose $U \subseteq V$.

($\Rightarrow$): Suppose U is a subspace. Then U must satisfy the 3 conditions above by the definition of a vector space.

($\Leftarrow$): Now suppose that U satisfies the 3 conditions above. The conditions (ii) and (iii) ensure that the restrictions of addition and scalar multiplication give well-defined binary operations

$$+ : U \times U \to U \qquad \mathbb{F} \times U \to U\,.$$

Since V is a vector space, then its addition is associative and commutative, so the same is true for U ((1) and (4)). Similarly, the distrubitive laws (7) and scalar multiplicativity (5) hold for U.

By (i), $0 \in U$ (2). Given $u \in U$, then

$$(-1)u = -u \in U$$

by (iii), so U has additive inverses (3). □

(0:30) Let's apply this criterion to some examples.

Example 1. • Let $V := \mathbb{F}^2$ and

$$U := \{(x_1, x_2) \in \mathbb{F}^2 \mid 3x_1 - x_2 = 0\}$$

[Check 3 conditions. Ask students what the zero vector is, how to start proofs of (ii) and (iii).]

- Let

$$V := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\},$$

which is a subset of $\mathbb{R}^{[0,1]}$.

- Let

$$V := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\},$$

which is a subset of $\mathbb{R}^{\mathbb{R}}$.

(0:40) II.3.1. *Sums of subspaces.*

It's sometimes useful to combine two subspaces V_1, V_2 of V. However, the union $V_1 \cup V_2$ is rarely itself a subspace. [Draw picture of example using $\mathbb{R}^2$ and coordinate axes as subspaces.] We instead can define their sum.

Definition 2. Given subspaces $V_1, \ldots, V_m$ of V, their *sum* is

$$V_1 + \cdots + V_m := \{v_1 + \cdots + v_m : v_1 \in V_1, \ldots, v_m \in V_m\},$$

i.e., the set of all possible sums of elements of $V_1, \ldots, V_m$.

Lemma 1. *With notation as above, $V_1 + \cdots + V_m$ is a subspace of V.*

Proof. Exercise. □

(0:45)

Example 2. Let $V := \mathbb{F}^4$,

$$U_1 := \{(s, s, t, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}$$
$$U_2 := \{(s, s, s, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}.$$

Q: Can we described $U_1 + U_2$?

Let's try computing systems of equations that describe U_1 and U_2. Given elements of $(a, a, b, b) \in U_1$ and $(c, c, c, d) \in U_2$, then

$$(a, a, b, b) + (c, c, c, d) = (a + c, a + c, b + c, b + d).$$

Since the first two coordinates are equal, we see that $U_1 + U_2 \subseteq W$, where

$$W := \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}.$$

We claim this is actually an equality. Given $(x, x, y, z) \in W$, we want to write it as a sum of an element of U_1 and an element of U_2. Goal:

$$(x, x, y, z) = (s, s, t, t) + (u, u, u, v)$$

for some $s, t, u, v \in \mathbb{F}$. Idea: Use this equation to derive some necessary conditions, then show that they are also sufficient. (Working backwards.) [Mention $P \implies Q$ vs. $Q \implies P$.]

Scratchwork: From the vector equation, we have

$$\begin{array}{ccccccc} s & & +u & & = & x \\ & t & +u & & = & y \\ & t & & +v & = & z \end{array}$$

which has augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & x \\ 0 & 1 & 1 & 0 & y \\ 0 & 1 & 0 & 1 & z \end{array}\right).$$

Row reducing, this is equivalent to

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & x-y+z \\ 0 & 1 & 0 & 1 & z \\ 0 & 0 & 1 & -1 & y-z \end{array}\right),$$

which corresponds to

$$\begin{array}{cccccc} s & & & +v & = & x-y+z \\ & t & & +v & = & z \\ & & u & -v & = & y-z\,. \end{array}$$

[Ask students how many solutions system has.] Since we only need one solution, we can set the free variable v to 0. And finally, we need to check that our answer actually works.

Claim: $W \subseteq U_1 + U_2$.

Proof. Given $(x,x,y,z) \in W$, then

$$(x,x,y,z) = (x-y+z, x-y+z, z, z) + (y-z, y-z, y-z, 0) \in U_1 + U_2\,.$$

□

Remark 2. Later we'll see an easy way to prove this containment using the notion of spanning sets.

(1:00)

Lemma 2. *Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the* smallest *[ask students](with respect to containment) subspace of V containing $V_1, \dots, V_m$. I.e., if W is another such subspace, then*

$$V_1 + \cdots + V_m \subseteq W\,.$$

Proof. Suppose W is another subspace containing $V_1, \dots, V_m$. Goal: $V_1 + \cdots + V_m \subseteq W$. Given $v \in V_1 + \cdots + V_m$, then

$$v = v_1 + \cdots + v_m$$

for some $v_1 \in V_1, \dots, v_m \in V_m$. Since $V_i \subseteq W$ for all $i = 1, \dots, m$, then $v_i \in W$ for all i. Since W is a subspace, then it is closed under addition, so

$$v_1 + \cdots + v_m \in W\,.$$

Thus $V_1 + \cdots + V_m \subseteq W$. □

(1:05)

II.3.2. *Direct sums.* In the example, an element of the sum $U_1 + U_2$ could be written as a sum of elements in U_1 and U_2 in infinitely many ways.

Q: Under what conditions is there only one way of writing an element of $V_1 + \cdots + V_m$ as such a sum?

Throughout this section, suppose $V_1, \ldots, V_m$ are subspaces of V.

Definition 3. The sum $V_1 + \cdots + V_m$ is a *direct sum* if each element of $V_1 + \cdots + V_m$ can be written *uniquely* as

$$v_1 + \cdots + v_m$$

with $v_i \in V_i$ for $i = 1, \ldots, m$. In this case, we write

$$V_1 \oplus \cdots \oplus V_m$$

to indicate that the sum is direct.

Example 3.

- Let $V := \mathbb{F}^3$ and consider the subspaces

$$V_1 := \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$$
$$V_2 := \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}.$$

  Exercise: Show $\mathbb{F}^3 = V_1 \oplus V_2$.
- In the example from the last section, we saw that $U_1 + U_2$ is *not* a direct sum.

Lemma 3. *$V_1 + \cdots + V_m$ is a direct sum iff the only way to write*

$$0 = v_1 + \cdots + v_m$$

with $v_i \in V_i$ for all i is by setting $v_i = 0$ for all i.

Proof. ($\Rightarrow$): Follow from the definition of direct sum, taking $v = 0$. ($\Leftarrow$): Given $v \in V_1 + \cdots + V_m$, suppose

$$v = v_1 + \cdots + v_m$$
$$v = u_1 + \cdots + u_m$$

where $u_i, v_i \in V_i$ for all $i = 1, \ldots, m$. Then

$$0 = v - v = (v_1 + \cdots + v_m) - (u_1 + \cdots + u_m) = \cdots = (v_1 - u_1) + \cdots + (v_m - u_m).$$

But we also have $0 = 0 + \cdots + 0$, so since this expression is unique, we must have $v_i - u_i = 0$, i.e., $v_i = u_i$ for all i. Thus the expression for v is unique. □

Proposition 2. *$V_1 + V_2$ is a direct sum iff $V_1 \cap V_2 = \{0\}$.*

Proof. Exercise. □

(1:20)

[Ask students what $U_1 \cap U_2$ was in first example, where

$$U_1 := \{(s, s, t, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}$$
$$U_2 := \{(s, s, s, t) \in \mathbb{F}^4 : s, t \in \mathbb{F}\}.$$

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