18.700 - LINEAR ALGEBRA, DAY 3
VECTOR SPACES

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CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will observe the algebraic properties of $\mathbb{R}^n$ and $\mathbb{C}^n$.
(2) Students will learn the definition of an abstract vector space.
(3) Students will see some examples of often-used vector spaces.
(4) Students will start or continue learning how to write proofs.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.
(1) Laptop + adapter (2) Worksheets

(0:00)

II. Lesson Plan

Announcements: • First pset due tomorrow. • Second pset posted later today.

II.1. **Last time.**

- Wrote down row reduction algorithm
- Applied algorithm to solve examples of linear systems
- Wrote solution sets in parametric form
- Characterized the possibilities for the number of solutions to linear systems (no solutions, unique solution, infinitely many solutions) in terms of the RREF of the corresponding augmented matrix

(0:05)

II.2. **1A: $\mathbb{R}^n$ and $\mathbb{C}^n$.** [Now following Axler, Chapter 1.]

The notion of a vector space was created to abstract many objects that share similar properties. The most classical example of vector spaces are $\mathbb{R}^2$ and $\mathbb{R}^3$, thought of as the real plane and real 3-space. We begin by generalizing these to higher dimensions.

II.2.1. *Definitions.*

Definition 1.

- Let $n \in \mathbb{Z}_{\geq 0}$ be a nonnegative integer. An *n-tuple* or *list* of *length n* is an ordered collection of n elements.
- Two lists are equal if they have the same length and the same entries in the same order.

Remark 1.

- Lists are usually written in the form $(z_1, z_2, \ldots, z_n)$. Note that lists must have **finite length**! The object $(z_1, z_2, \ldots)$ is not a list, but rather an infinite sequence.
- Unlike sets, the order and multiplicities of elements in tuples matters! So $\{2,3\} = \{3,2\}$ and $\{4,4,4\} = \{4\}$, but $(2,3) \neq (3,2)$ and $(4,4,4) \neq (4)$.

For the rest of the lecture, fix $n \in \mathbb{Z}_{>0}$.

Definition 2. Let $\mathbb{F}^n$ be the set of all n-tuples with entries in $\mathbb{F}$:

$$\mathbb{F}^n := \{(x_1, \ldots, x_n) : x_k \in \mathbb{F} \text{ for all } k = 1, \ldots, n\}.$$

Given $(x_1, \ldots, x_n) \in \mathbb{F}^n$, its k^{th} *coordinate* or *entry* is x_k.

Remark 2. It is sometimes convenient to write elements of $\mathbb{F}^n$ vertically, as column vectors.

Example 1. $\mathbb{C}^{17} = \{(z_1, \ldots, z_{17}) : z_1, \ldots, z_{17} \in \mathbb{C}\}$. [Can't visualize it, but still makes sense algebraically.]

(0:15)

II.2.2. *Algebraic properties of* $\mathbb{F}^n$.

(1) Addition operation
(2) 0-vector
(3) Additive inverses
(4) Scalar multiplication
(5) Commutativity of addition

[Draw examples in $\mathbb{R}^2$ to illustrate geometric intuition behind (1), (3), and (4). Prove commutativity of addition.]

(0:25)

II.3. **Abstract vector spaces.** Now that we've observed some of the algebraic properties of $\mathbb{F}^n$, we're going to give an abstract definition of a vector space as something with these properties, i.e.,a set equipped with addition and scalar multiplication operations satisfying some conditions.

Definition 3. A *vector space* is a set V equipped with an addition operation

$$\begin{aligned} + : V \times V &\to V \\ (u, v) &\mapsto u + v \end{aligned}$$

and a scalar multiplication operation

$$\begin{aligned} \mathbb{F} \times V &\to V \\ (\lambda, v) &\mapsto \lambda v \end{aligned}$$

satisfying the following properties.

(1) (Associativity of addition):

$$(u + v) + w = u + (v + w)$$

for all $u, v, w \in V$.

(2) (Additive identity): There exists an element $0 \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.

(3) (Additive inverses): For each $v \in V$, there exists $w \in V$ such that $v + w = w + v = 0$.

(4) (Commutative of addition): $u + v = v + u$ for all $u, v \in V$.

(5) (Scalar multiplicative identity): $1\,v = v$ for all $v \in V$.

(6) (Associativity of scalar multiplication): $a(bv) = (ab)v$ for all $a, b \in \mathbb{F}$ and $v \in V$.

(7) (Distributive laws): [ask students]

   (a) $a(u + v) = au + av$ for all $a \in \mathbb{F}$ and all $u, v, \in V$.

   (b) $(a + b)v = av + bv$ for all $a, b \in F$ and all $v \in V$.

Elements of a vector space are called *vectors*. [Tell engineer, physicist, mathematician joke.]

Remark 3. Structures satisfying (1), (2), (3) are called *groups*; structures satisfying (1), (2), (3), (4) are called *abelian groups*.

Remark 4. Elements of $\mathbb{F}$ are sometimes called *scalars*. When we need to specify the underlying field, we will say that V is a *vector space over* $\mathbb{F}$ or an $\mathbb{F}$-*vector space*.

(0:40) **Example 2.**

- $\mathbb{R}^2$ is a vector space over $\mathbb{R}$.
- The *zero vector space* or *trivial vector space* is the set $\{0\}$ consisting of just the zero vector. Exercise: Check that it satisfies the axioms of a vector space.
- Let

$$\mathbb{F}^\infty := \{(x_1, x_2, \ldots) : x_k \in \mathbb{F} \text{ for all } k = 1, 2, \ldots\},$$

with addition and scalar multiplication defined componentwise:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = \cdots$$
$$\lambda(x_1, x_2, \ldots) = \cdots .$$

Exercise: Check that $\mathbb{F}^\infty$ satisfies the axioms of a vector space. What is the zero vector?

- Let S be any set. Define

$$\mathbb{F}^S := \{f : S \to \mathbb{F}\}$$

with addition and scalar multiplication defined pointwise, i.e., for any $f, g \in \mathbb{F}^S$ and any $\lambda \in \mathbb{F}$

$$(f+g)(x) := f(x) + g(x)(\lambda f)(x) := \lambda f(x)$$

for all $x \in S$.

Exercise: $\mathbb{F}^S$ is a vector space.

Remark 5. One can view $\mathbb{F}^n$ as $\mathbb{F}^S$ where $S = \{1, 2, \ldots, n\}$.

$$\mathbb{F}^S \longleftrightarrow \mathbb{F}^n$$
$$(f : S \to \mathbb{F}) \longmapsto (f(1), f(2), \ldots, f(n))$$
$$(g : S \to \mathbb{F}) \longleftarrow (x_1, x_2, \ldots, x_n)$$
$$k \mapsto x_k$$

One can similarly view $\mathbb{F}^\infty$ as $\mathbb{F}^S$ where $S = \mathbb{Z}_{>0} = \{1, 2, \ldots\}$. Later we'll give a more formal definition of when two vector spaces are "the same"; the precise term is *isomorphic*.

(0:50) Henceforth, unless otherwise mentioned, V is a vector space over $\mathbb{F}$.

Lemma 1. *Every vector space has a* unique *additive identity.*

Proof. Let V be a vector space and suppose $0, 0' \in V$ are both additive identity elements. Then [start in the middle]

$$0 = 0 + 0' = 0';$$

the first equality follows because $0'$ is an additive identity, and the second because 0 is. □

Lemma 2. *Every element of a vector space has a unique additive inverse.*

Proof. Worksheet. □

Since additive inverses are unique, then we can unambiguously write $-v$:

Definition 4. Given $v, w \in V$, we define:

- $-v$ to be the additive inverse of v;
- $w - v := w + (-v)$.

Lemma 3.

- *For all $v \in V$, we have $0\,v = 0$.*
- *For all $a \in \mathbb{F}$, we have $a \cdot 0 = 0$.*

[Remark on 0 the scalar vs 0 the vector.]

Proof. Worksheet. □

Lemma 4. *For all $v \in V$, we have $(-1)v = -v$.*

Proof. Given $v \in V$, then

$$v + (-1)v = 1\,v + (-1)v = (1 + (-1))v = 0\,v = 0\,.$$

□

(1:00) II.4. **Worksheet.**

II.5. **Subspaces.** Given a vector space V, it is often useful to consider smaller vector spaces that are contained inside V. We call these subspaces. [Draw picture of a line
(1:10) through the origin inside $\mathbb{R}^2$.]

Definition 5. A subset $U \subseteq V$ is called a *(vector) subspace* if U is also a vector space when considered with the same addition, scalar multiplication, and additive identity as V.

Remark 6.

- Also sometimes known as *linear subspaces*.
- $\{0\}$ and V itself are always subspaces of a vector space V.

To check if a subset $U \subseteq V$ is a subspace, it's not necessary to check all the vector space axioms. We get some for free from knowing already that V is a vector space.

Proposition 1 (Subspace criterion)**.** *A subset $U \subseteq V$ is a subspace of V iff U satisfies the following.*

(i) $0 \in U$.
(ii) U *is closed under addition, i.e., if* $u, v \in U$, *then* $u + v \in U$.
(iii) U *is closed under scalar multiplication, i.e., if* $a \in \mathbb{F}$ *and* $u \in U$, *then* $au \in U$.

Proof. Suppose $U \subseteq V$.
(⇒): Suppose U is a subspace. Then U must satisfy the 3 conditions above by the definition of a vector space.
(⇐): Now suppose that U satisfies the 3 conditions above. The conditions (ii) and (iii) ensure that the restrictions of addition and scalar multiplication give well-defined binary operations

$$+ : U \times U \to U \qquad \mathbb{F} \times U \to U\,.$$

Since V is a vector space, then its addition is associative and commutative, so the same is true for U ((1) and (4)). Similarly, the distrubitive laws (7) and scalar multiplicativity (5) hold for U.
By (i), $0 \in U$ (2). Given $u \in U$, then

$$(-1)u = -u \in U$$

by (iii), so U has additive inverses (3). □

(1:20) Let's apply this criterion to some examples.

Example 3.
- Let $V := \mathbb{F}^2$ and
$$U := \{(x_1, x_2) \in \mathbb{F}^2 \mid 3x_1 - x_2 = 0\}$$
[Check 3 conditions. Ask students what the zero vector is, how to start proofs of (ii) and (iii).]
- Let
$$V := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous}\},$$
which is a subset of $\mathbb{R}^{[0,1]}$.
- Let
$$V := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable}\},$$
which is a subset of $\mathbb{R}^{\mathbb{R}}$.

(1:30) II.5.1. *Sums of subspaces.*

It's sometimes useful to combine two subspaces V_1, V_2 of V. However, the union $V_1 \cup V_2$ is rarely itself a subspace. [Draw picture of example using $\mathbb{R}^2$ and coordinate axes as subspaces.] We instead can define their sum.

Definition 6. Given subspaces $V_1, \dots, V_m$ of V, their *sum* is
$$V_1 + \cdots + V_m := \{v_1 + \cdots + v_m : v_1 \in V_1, \dots, v_m \in V_m\},$$
i.e., the set of all possible sums of elements of $V_1, \dots, V_m$.

Lemma 5. *With notation as above, $V_1 + \cdots + V_m$ is a subspace of V.*

Proof. Exercise. □