18.700 - LINEAR ALGEBRA, DAY 24
BILINEAR AND MULTILINEAR FORMS
DETERMINANTS

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CONTENTS

I. PRE-CLASS PLANNING

I.1. Goals for lesson.

(1) Students will learn the definitions of an inversion and the sign of a permutation.
(2) Students will learn how permuting the entries affects the value of an alternating multilinear form.
(3) Students will learn properties of the determinant, such as multiplicativity, interpretation as product of eigenvalues, etc.

I.2. Methods of assessment.

(1) Student responses to questions posed during lecture
(2) Student responses to worksheet

I.3. Materials to bring.

(1) Laptop + adapter (2) Worksheets (3) Chalk

(0:00)

II. Lesson Plan

II.1. **Last time.**

- Proved that every linear map has Jordan basis.
- Defined trace: $\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}$.
- Proved properties of trace. [Ask students about trace and eigenvalues.]
- Defined multilinear forms.

II.2. **Worksheet.**

II.3. **9B: Multilinear forms.** Notation: $V^m = \overbrace{V \times \cdots \times V}^{m \text{ times}}$. So an element of V^m is of the form $(v_1, \ldots, v_m)$ with $v_i \in V$ for all i.

- An *m-linear form* on V is a function $\beta : V^m \to \mathbb{F}$ that is linear in each component when the others are held fixed. I.e., for each $k \in \{1, \ldots, m\}$ and $u_1, \ldots, u_m \in V$, the map
$$\begin{aligned} V &\to \mathbb{F} \\ v &\mapsto \beta(u_1, \ldots, u_{k-1}, v, u_{k+1}, \ldots, u_m) \end{aligned}$$
is linear.
- An m-linear form $\alpha \in V^{(m)}$ is *alternating* if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_j = v_k$ for some $j, k \in \{1, \ldots, m\}$ with $j \neq k$.

Lemma 1. *$V^{(m)}$ is a vector space, and $V^{(m)}_{\text{alt}}$ is a subspace.*

Proof. Exercise. □

Lemma 2. *Let $\alpha \in V^{(m)}_{\text{alt}}$. If $v_1, \ldots, v_m \in V$ is linearly dependent, then*
$$\alpha(v_1, \ldots, v_m) = 0\,.$$

Proof idea. Use the Linear Dependence Lemma to express v_k as a linear combination of the others. Then use multilinearity and alternating property. Details left as an exercise. □

Proposition 3. *Let $\alpha \in V^{(m)}_{\text{alt}}$ and $v_1, \ldots, v_m \in V$. Swapping the vectors in any two slots of $\alpha(v_1, \ldots, v_m)$ changes the value by a factor of -1.*

Proof idea. For simplicity, suppose $m = 2$. Then
$$0 = \alpha(v + w, v + w) = \cancelto{0}{\alpha(v, v)} + \alpha(v, w) + \alpha(w, v) + \cancelto{0}{\alpha(w, w)}.$$
The proof is virtually the same for $m \geq 2$. □

Q: What if we perform multiple swaps? For example, suppose that $\alpha \in V^{(3)}_{\text{alt}}$ and $v_1, v_2, v_3 \in V$. Then
$$\alpha(v_3, v_1, v_2) = -\alpha(v_1, v_3, v_2) = \alpha(v_1, v_2, v_3)\,.$$
This leads us to investigate more general permutations.

Definition 4. Let $m \in \mathbb{Z}_{>0}$.

- A *permutation* of $(1, \ldots, m)$ is a rearrangement of $(1, \ldots, m)$, i.e., a list $(j_1, \ldots, j_m)$ that contains each of $1, \ldots, m$ exactly once.
- Denote the set of all permutations of $(1, \ldots, m)$ by $\operatorname{perm}(m)$ or S_m. (The *symmetric group* on $1, \ldots, m$.)

Example 5. $(2, 1, 4, 3) \in \operatorname{perm}(4)$.

Definition 6. Suppose $(j_1, \ldots, j_m) \in \operatorname{perm}(m)$.

- An *inversion* of $(j_1, \ldots, j_m)$ is a pair of integers (k, ℓ) with $k, \ell \in \{1, \ldots, m\}$ such that $k < \ell$ and k appears *after* ℓ in the list $(j_1, \ldots, j_m)$.
- Let N be the number of inversions of $(j_1, \ldots, j_m)$. The *sign* of $(j_1, \ldots, j_m)$ is

$$\operatorname{sgn}(j_1, \ldots, j_m) := (-1)^N .$$

Example 7.

- Consider $(2, 1, 3, 4) \in \operatorname{perm}(4)$. It has exactly one inversion, namely $(1, 2)$, so it has sign $(-1)^1 = -1$.
- The permutation $(1, \ldots, m)$ has no inversions (the numbers are all in increasing order), so it has sign $(-1)^0 = 1$.
- Consider the permutation $(2, 3, \ldots, m, 1)$. Its inversions are

$$(1, 2), (1, 3), \ldots, (1, m)$$

  so it has sign $(-1)^{m-1}$.

Proposition 8. *Swapping two entries in a permutation multiplies the sign of the permutaiton by* -1.

Proof sketch. [Shorten or skip, if necessary.] Let π be the original permutation, and π' be the permutation obtained from swapping the i^{th} and j^{th} entries of π. Denote the i^{th} entry of π by $\pi(i)$. Then $\pi(i) < \pi(j)$ iff $\pi'(i) > \pi'(j)$, so we have either added or subtracted exactly 1 inversion so far.

Consider the entries not in between the i^{th} and j^{th} spots. For these entries, there is no change in whether they were in order or not. [Draw picture.]

Now consider $\pi(k)$ with $i < k < j$.

<u>Case 1</u>: $\pi(k)$ was in order with respect to both $\pi(i)$ and $\pi(j)$, i.e., $\pi(i) < \pi(k) < \pi(j)$. Then

$$\pi'(i) > \pi'(k) > \pi'(j)$$

so we have 2 more inversions, multiplying the sign by $(-1)^2 = 1$.

<u>Case 2</u>: $\pi(i) > \pi(k) > \pi(j)$. Similar.

<u>Case 3</u>: $\pi(i) < \pi(k)$ and $\pi(k) > \pi(j)$. Then

$$\pi'(i) = \pi(j) < \pi(k) = \pi'(k)$$
$$\pi'(k) = \pi(k) > \pi(i) = \pi'(j)$$

so we have the same number of inversions that we started with, and the sign is unchanged.

<u>Case 4</u>: $\pi(i) > \pi(k)$ and $\pi(k) < \pi(j)$. Similar.

Thus in all cases we have an odd number of inversions, so $\operatorname{sign}(\pi') = -\operatorname{sign}(\pi)$. $\square$

Proposition 9. *Suppose $m \in \mathbb{Z}_{>0}$ and $\alpha \in V_{\text{alt}}^{(m)}$. Then*

$$\alpha(v_{j_1}, \ldots, v_{j_m}) = \operatorname{sign}(j_1, \ldots, j_m)\alpha(v_1, \ldots, v_m)$$

Proof idea. We can get from $(j_1, \ldots, j_m)$ to $(1, \ldots, m)$ by a series of swaps. Each swap changes the sign of α by a factor of -1, and also changes the sign of the remaining permutation by a factor of -1. □

Theorem 10. *Let $n := \dim(V)$. Suppose $e_1, \ldots, e_n$ is a basis of V. Suppose $v_1, \ldots, v_n \in V$. For each k, write*

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

for some $b_{1,k}, \ldots, b_{n,k} \in \mathbb{F}$. Then

$$\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} \operatorname{sign}(j_1, \ldots, j_n) b_{j_1,1} \cdots b_{j_n,n}$$

for all $\alpha \in V_{\text{alt}}^{(n)}$.

Proof.

$$\begin{aligned}
\alpha(v_1, \ldots, v_n) &= \alpha\left(\sum_{j_1=1}^{n} b_{j_1,1} e_{j_1}, \ldots \sum_{j_n=1}^{n} b_{j_n,1} e_{j_n}\right) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} b_{j_1,1} \cdots b_{j_n,n} \alpha(e_{j_1}, \ldots, e_{j_n}) \\
&= \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} b_{j_1,1} \cdots b_{j_n,n} \alpha(e_{j_1}, \ldots, e_{j_n}) \\
&= \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} b_{j_1,1} \cdots b_{j_n,n} \operatorname{sign}(j_1, \ldots, j_n) \alpha(e_1, \ldots, e_n) \\
&= \alpha(e_1, \ldots, e_n) \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} \operatorname{sign}(j_1, \ldots, j_n) b_{j_1,1} \cdots b_{j_n,n},
\end{aligned}$$

where the third equality holds because $\alpha(e_{j_1}, \ldots, e_{j_n}) = 0$ if $j_1, \ldots, j_n$ are not distinct. □

Corollary 11. $\dim(V_{\text{alt}}^{(n)}) = 1$.

Proof. Let $n := \dim(V)$. Suppose $\alpha, \alpha' \in V_{\text{alt}}^{(n)}$ with $\alpha \neq 0$. Then $\alpha(e_1, \ldots, e_n) \neq 0$ for some $e_1, \ldots, e_n \in V$. Then $e_1, \ldots, e_n$ is linearly independent (contrapositive of earlier result). Let

$$c := \frac{\alpha'(e_1, \ldots, e_n)}{\alpha(e_1, \ldots, e_n)}.$$

Letting $b_{j,k}$ be as above, then

$$\begin{aligned}
\alpha'(v_1, \ldots, v_n) &= \alpha'(e_1, \ldots, e_n) \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} \operatorname{sign}(j_1, \ldots, j_n) b_{j_1,1} \cdots b_{j_n,n} \\
&= c\alpha(e_1, \ldots, e_n) \sum_{(j_1,\ldots,j_n) \in \operatorname{perm}(n)} \operatorname{sign}(j_1, \ldots, j_n) b_{j_1,1} \cdots b_{j_n,n} \\
&= c\alpha(v_1, \ldots, v_n).
\end{aligned}$$

Thus $\alpha' = c\alpha$. Thus $\dim(V_{\text{alt}}^{(n)}) \leq 1$.

It remains to show that $\dim(V_{\text{alt}}^{(n)}) = 1$. For details, see 9.37 in the text book. (Uses the concept of the dual basis of a basis.) $\square$

[Skip if necessary.] To prove that $\dim(V_{\text{alt}}^{(m)}) = 1$, we will need some more results on linear functionals. Recall that $V^\vee$, the dual space, is

$$V^\vee = \mathcal{L}(V, \mathbb{F}) = \{\varphi : V \to \mathbb{F} \mid \varphi \text{ is linear}\}.$$

Fix $j \in \{1, \dots, n\}$. Define

$$\begin{aligned} \varphi_j : \mathbb{F}^n &\to \mathbb{F} \\ (x_1, \dots, x_n) &\mapsto x_j , \end{aligned}$$

i.e., projection onto the j^{th} coordinate. Then φ_j is linear and

$$\varphi_j(e_k) = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{otherwise.} \end{cases}$$

We can define a similar notion in general.

Definition 12. Let $\mathcal{B} := (v_1, \dots, v_n)$ be a basis of V. The *dual basis of $\mathcal{B}$* is the list $\mathcal{B}^\vee := (\varphi_1, \dots, \varphi_n)$ in $V^\vee$, where φ_j is defined by

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 13. *Suppose V is finite-dimensional. Then $\mathcal{B}$ is a basis of $V^\vee$.*

Remark 14. φ_j is sometimes denoted $v_j^\vee$.

Proof, completed. Let $e_1, \dots, e_n \in V$ be a basis and let $\varphi_1, \dots, \varphi_n \in V^\vee$ be its dual basis. Define

$$\alpha(v_1, \dots, v_n) = \sum_{(j_1,\dots,j_n) \in \text{perm}(n)} \text{sign}(j_1, \dots, j_n)\varphi_{j_1}(v_1) \cdots , \varphi_{j_n}(v_n).$$

One can show that α is n-linear, alternating, and nonzero. Indeed,

$$\alpha(e_1, \dots, e_n) = 1$$

since only the term corresponding to the permutation $(1, \dots, n)$ is nonzero. $\square$

Proposition 15. *Let $n := \dim(V)$. Suppose $\alpha \in V_{\text{alt}}^{(n)}$ is nonzero and $e_1, \dots, e_n \in V$. Then $e_1, \dots, e_n$ is linearly independent iff*

$$\alpha(e_1, \dots, e_n) \neq 0.$$

Proof. ($\Leftarrow$): Assume $\alpha(e_1, \dots, e_n) \neq 0$. Then $e_1, \dots, e_n$ is linearly independent: this is the converse of an earlier result.

($\Rightarrow$): Assume $e_1, \dots, e_n$ is linearly independent. Since $n = \dim(V)$, then it is a basis of V. Since $\alpha \neq 0$ then there exist $v_1, \dots, v_n \in V$ such that $\alpha(v_1, \dots, v_n) \neq 0$. Since $e_1, \dots, e_n$ is a basis, then for each k,

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

for some $b_{1,k}, \dots, b_{n,k} \in \mathbb{F}$. By an earlier result,

$$0 \neq \alpha(v_1, \dots, v_n) = \alpha(e_1, \dots, e_n) \sum_{(j_1,\dots,j_n) \in \mathrm{perm}(n)} \mathrm{sign}(j_1, \dots, j_n) b_{j_1,1} \cdots b_{j_n,n} .$$

Thus $\alpha(e_1, \dots, e_n) \neq 0$. □

II.4. **9C: Determinants.** [Show 3Blue1Brown video: `https://www.youtube.com/watch?v=Ip3X9LOh2dk` beginning until about 5:30 or so.]

Definition 16 ("Moral" definition)**.** Suppose $T \in \mathcal{L}(V)$. Consider the unit hypercube C formed by the standard basis $e_1, \dots, e_n$. Then the image of C under T is a parallelotope P. The determinant of T is the signed volume of P, where the sign keeps track of the orientation of $T(e_1), \dots, T(e_n)$.

Definition 17. Suppose $m \in \mathbb{Z}_{>0}$ and $T \in \mathcal{L}(V)$. Define the map

$$\begin{aligned} T^\# : V_{\mathrm{alt}}^{(m)} &\to V_{\mathrm{alt}}^{(m)} \\ \alpha &\mapsto T^\#(\alpha) \end{aligned}$$

where

$$T^\#(\alpha)(v_1, \dots v_m) := \alpha(T(v_1), \dots, T(v_m)) .$$

Remark 18. The textbook instead denotes $T^\#(\alpha)$ by α_T.

Let $n := \dim(V)$. Recall that $V_{\mathrm{alt}}^{(n)}$ has dimension 1. Any linear map from a 1-dimensional space to itself is multiplication by a scalar.

Definition 19.

- Let $n := \dim(V)$. Suppose $T \in \mathcal{L}(V)$. The *determinant of* T, denoted $\det(T)$, is the unique scalar in $\mathbb{F}$ such that

  $$T^\#(\alpha) = \det(T)\alpha$$

  for all $\alpha \in V_{\mathrm{alt}}^{(n)}$.
- Suppose $A \in M_{n \times n}(\mathbb{F})$. Define $\det(A) := \det(L_A)$ where L_A is the left multiplication map

  $$\begin{aligned} L_A : \mathbb{F}^n &\to \mathbb{F}^n \\ v &\mapsto Av . \end{aligned}$$

Example 20.

- Consider the identity operator I.

  $$I^\#(\alpha)(v_1, \dots, v_n) = \alpha(Iv_1, \dots, Iv_n) = \alpha(v_1, \dots, v_n) ,$$

  so $I^\#(\alpha) = \alpha$. Thus [ask students] $\det(I) = 1$.
- Suppose T is diagonalizable with a basis of eigenvectors $v_1, \dots, v_n$ with eigenvalues $\lambda_1, \dots, \lambda_n$. Given $0 \neq \alpha \in V_{\mathrm{alt}}^{(n)}$, then

$$T^\#(\alpha)(v_1, \dots, v_n) = \alpha(T(v_1), \dots, T(v_n)) = \alpha(\lambda_1 v_1, \cdots, \lambda_n v_n) = \lambda_1 \cdots \lambda_n \alpha(v_1, \dots, v_n) .$$

  Thus $\det(T) = \lambda_1 \cdots \lambda_n$, the product of the eigenvalues of T.

Theorem 21. *Suppose $n \in \mathbb{Z}_{>0}$. The map*

$$\varphi : \overbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}^{n \text{ times}} \to \mathbb{F}$$

$$(v_1, \ldots, v_n) \mapsto \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

is an alternating n-linear form on $\mathbb{F}^n$.

Proof. Let $\mathcal{E} := (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{F}^n$. Suppose $v_1, \ldots, v_n \in \mathbb{F}^n$. Define

$$T : \mathbb{F}^n \to \mathbb{F}^n$$

$$e_i \mapsto v_i$$

for each $i = 1, \ldots, n$. Then

$$[T]_{\mathcal{E}} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} .$$

Let α be an alternating n-linear form on $\mathbb{F}^n$ such that $\alpha(e_1, \ldots, e_n) = 1$. Then

$$\begin{aligned} \psi(v_1, \ldots, v_n) &= \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \det(T) = \det(T)\alpha(e_1, \ldots, e_n) \\ &= \alpha(T(e_1), \ldots, T(e_n)) = \alpha(v_1, \ldots, v_n) . \end{aligned}$$

Thus $\psi = \alpha$, so the map ψ is an alternating n-linear form. □

Remark 22. One often considers the determinant of a square matrix A as a multilinear form on the columns of A.

Corollary 23. *Suppose $n \in \mathbb{Z}_{>0}$ and $A \in M_{n \times n}(\mathbb{F})$. Then*

$$\det(A) = \sum_{(j_1, \ldots, j_n) \in \operatorname{perm}(n)} \operatorname{sign}(j_1, \ldots, j_n) A_{j_1,1} \cdots A_{j_n,n} .$$

Proof. Letting $e_1, \ldots, e_n$ be the standard basis for $\mathbb{F}^n$, note that

$$\det \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix} = \det(I) = 1 .$$

The result the follows from an earlier result on alternating n-linear forms on an n-dimensional vector space. □

Example 24. If A is 2×2, then $\operatorname{perm}(2) = \{(1, 2), (2, 1)\}$, so [ask students]

$$\det(A) = A_{1,1}A_{2,2} + (-1)A_{2,1}A_{1,2} .$$

Remark 25. While this formula can be used for small matrices, the sum contains $n!$ terms, which grows rapidly as n increases.

Proposition 26 (The determinant is multiplicative)**.**

- $\det(ST) = \det(S)\det(T)$ *for all $S, T \in \mathcal{L}(V)$.*

- $\det(AB) = \det(A)\det(B)$ *for all* $A, B \in M_{n\times n}(\mathbb{F})$.

Proposition 27. *$T \in \mathcal{L}(V)$ is invertible iff $\det(T) \neq 0$. In this case, $\det(T^{-1}) = \dfrac{1}{\det(T)}$.*

Proof. ($\Rightarrow$): Assume T is invertible. Then

$$1 = \det(I) = \det(TT^{-1}) = \det(T)\det(T^{-1})$$

Thus $\det(T) \neq 0$.

($\Leftarrow$): Exercise. (Usually proved using Cramer's rule.) □

Proposition 28. *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then λ is an eigenvalue of T iff $\det(\lambda I - T) = 0$.*

Proof. λ is an eigenvalue of T iff $T - \lambda I$ is not invertible iff $\lambda I - T = -(T - \lambda I)$ is not invertible iff $\det(\lambda I - T) = 0$ by the previous result. □

Proposition 29. *Suppose $T \in \mathcal{L}(V)$ and $S : W \to V$ is an invertible linear map. Then*

$$\det(S^{-1}TS) = \det(T)\,.$$

Proof. [Skip if necessary. See 9.52 in the textbook.] Since S is an isomorphism, then $\dim(W) = \dim(V)$; call this common value n. Given $\tau \in W^{(n)}_{\text{alt}}$, let $\alpha := (S^{-1})^{\#}(\tau)$, so

$$\alpha(v_1, \ldots, v_n) = \tau(S^{-1}v_1, \ldots, S^{-1}v_n)$$

for all $v_1, \ldots, v_n \in V$. Given $w_1, \ldots, w_n \in W$, then

$$\begin{aligned}(S^{-1}TS)^{\#}(\tau)(w_1, \ldots, w_n) = \tau(S^{-1}TSw_1, \ldots, S^{-1}TSw_n) &= \overbrace{(S^{-1})^{\#}(\tau)}^{\alpha}(TSw_1, \ldots, TSw_n)\\ &= T^{\#}(\alpha)(Sw_1, \ldots, Sw_n) = \det(T)\alpha(Sw_1, \ldots, Sw_n)\\ &= \det(T)S^{\#}(\alpha)(w_1, \ldots, w_n) = \det(T)\tau(w_1, \ldots, w_n).\end{aligned}$$

□

Proposition 30. *Suppose $T \in \mathcal{L}(V)$ and $\mathcal{B} := (v_1, \ldots, v_n)$ a basis of V. Then*

$$\det(T) = \det([T]_{\mathcal{B}})\,.$$

Proof. Let $\mathcal{E} = (e_1, \ldots, e_n)$ be the standard basis of $\mathbb{F}^n$, and let $S : \mathbb{F}^n \to V$ be the isomorphism that takes $S(v_i) = e_i$ for each i. Then ${}_{\mathcal{B}}[S]_{\mathcal{E}} = I$, so

$${}_{\mathcal{E}}[S^{-1}TS]_{\mathcal{B}} = {}_{\mathcal{E}}[S^{-1}]_{\mathcal{E}}\ [T]_{\mathcal{B}}\ {}_{\mathcal{B}}[S]_{\mathcal{E}} = I[T]_{\mathcal{B}}I = [T]_{\mathcal{B}}\,.$$

Then

$$\det(T) = \det(S^{-1}TS) = \det\left([S^{-1}TS]_{\mathcal{E}}\right) = \det\left([T]_{\mathcal{B}}\right)\,.$$

□

Theorem 31 (Determinant is product of eigenvalues)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, repeated with algebraic multiplicity. Then*

$$\det(T) = \lambda_1 \cdots \lambda_n\,.$$

Proof. By a previous result, there exists a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is upper triangular with the eigenvalues of T on its diagonal. Then

$$\det(T) = \det([T]_{\mathcal{B}}) = \lambda_1 \cdots \lambda_n$$

□

Corollary 32. *Suppose $\mathbb{F} = \mathbb{C}$ and $T = \mathcal{L}(V)$. Then*

$$\det(zI - T) = \operatorname{charpoly}(T)$$

considered as polynomial functions of $z \in \mathbb{C}$.

We now use this to redefine the notion of characteristic polynomial, extending the definition to operators on $\mathbb{R}$-vector spaces.

Definition 33. Let V be a finite-dimensional $\mathbb{F}$-vector space and $T \in \mathcal{L}(V)$. Then polynomial defined by

$$z \mapsto \det(zI - T)$$

is the *characteristic polynomial of T*.

Proposition 34.

(a) Suppose A is a square matrix. Then $\det(A^t) = \det(A)$.
(b) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Then

$$\det(T^*) = \overline{\det(T)}\,.$$

Proof. (a) Let $n \in \mathbb{Z}_{>0}$. Define

$$\alpha : (\mathbb{F}^n)^n \to \mathbb{F}$$
$$(v_1, \ldots, v_n) \mapsto \det \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}.$$

From the permutation formula for the determinant, then α is an n-linear form on $\mathbb{F}^n$.

Claim: α is alternating. Suppose $v_1, \ldots, v_n \in \mathbb{F}^n$ with $v_j = v_k$ for some $j \neq k$. Then the matrix

$$B := \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

has rank $< n$. Since row rank = column rank, then B^t has rank $< n$, so B^t is not invertible. Then the columns of B^t are linearly dependent, hence

$$\alpha(v_1, \ldots, v_n) = \det(B^t) = 0\,.$$

Thus α is alternating.

Since $\alpha, \det \in V_{\text{alt}}^{(n)}$ and this space is 1-dimensional, then $\alpha = c \det$ for some $c \in \mathbb{F}$. Applying α to the standard basis, we have

$$\alpha(e_1, \ldots, e_n) = \det(I^t) = \det(I) = 1\,,$$

so $c = 1$ and $\alpha = \det$ (thought of as a function of the columns).

(b) Exercise.

□

Proposition 35. *Let A be a square matrix.*

(a) If any two columns of A are equal, then $\det(A) = 0$.
(b) Let B be the matrix obtained by swapping two rows of A. Then $\det(B) = -\det(A)$.
(c) If B is obtained by multiplying a column of A by a scalar c, then $\det(B) = c\det(A)$.
(d) If B is obtained by adding a scalar multiple of one column of A to another, then $\det(B) = \det(A)$.
(e) In all the above, the word "column" may be replaced by the word "row".

Proof. Straightforward. Last statement holds because $\det(A^t) = \det(A)$ by the previous result. For instance,

$$\det(v_1 + cv_2, v_2, \ldots, v_n) = \det(v_1, v_2, \ldots, v_n) + c\underbrace{\det(v_2, v_2, \ldots, v_n)}_{0} = \det(v_1, \ldots, v_n)\,.$$

□

Remark 36. This provides a strategy for evaluating the determinant of a matrix A:

(1) Perform row operations on A to obtain an upper triangular matrix, keeping track of any swaps or rescalings of rows.
(2) The determinant of the resulting upper triangular matrix is simply the product of the diagonal entries. Multiply this by the factors from swaps and rescalings.

Example 37. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\,.$$

Then $\det(A) = \cdots = 2$. [Row reduce matrix.]